TORIC ACTIONS ON b-SYMPLECTIC MANIFOLDS

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Abstract. We study Hamiltonian actions on b-symplectic manifolds with a focus on the effective case of half the dimension of the manifold. In particular, we prove a Delzant-type theorem that classifies these manifolds using polytopes that reside in a certain enlarged and decorated version of the dual of the Lie algebra of the torus. At the end of the paper we suggest further avenues of study, including an example of a toric action on a b^2-manifold and applications of our ideas to integrable systems on b-manifolds.

1. Introduction

The role of symmetries in reducing the number of variables in a Hamiltonian system has motivated symplectic geometers to study Hamiltonian Lie group actions, moment maps, and symplectic reduction. One of the great fruits of this research is an understanding of the correspondence between geometric properties of a Hamiltonian system and combinatorial properties of the image of its moment map.

It is a well-known fact that the image of the moment map of a compact Lie group action on a compact symplectic manifold is *convex* under the conditions specified in [GS1, GS2, K] (see also [Sj] for some extensions to the non-compact setting). Furthermore if the group acting on the manifold is a torus of half the dimension of the manifold and the action is effective, the image of the moment map is a rational simple polytope (called a *Delzant* polytope) and one can reconstruct the symplectic manifold from this polytope via symplectic reduction [D].

The authors of [LT] extend this result to the *singular* case, proving a correspondence between symplectic toric orbifolds and Delzant polytopes with positive integer labels on each facet. Another singular context to which this extension can be performed is that of Poisson manifolds. A Poisson structure endows a manifold with a natural foliation by symplectic leaves

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which are preserved under Hamiltonian actions of the manifold. One could therefore study the properties of the image of the moment map as a moduli problem on the space of leaves of the symplectic foliation. This is a delicate matter since these leaves will usually have different dimension.

Extensions of the convexity and Delzant's theorem to the Poisson category are scarce (see [Z] for an account). In this paper we prove a Delzant theorem for a class of Poisson manifold which is close to the symplectic class called *b-symplectic manifolds*. These objects were first studied as manifolds with boundary in the works of Melrose [Me] and Nest and Tsygan [NT]; recent treatments of the subject (in [GMP1] and [GMP2]) study the objects as manifolds with distinguished hypersurfaces.

The symplectic groupoids integrating b-manifolds have been lately considered by Gualtieri and Li in [GL]. The topology of these manifolds have been studied further in [C], [FMM] and [MO1, MO2]. The class of b-symplectic manifolds that we consider are compact b-symplectic manifolds with the property that the induced symplectic foliation on the critical hypersurface has compact leaves (see [GMP1]). The critical hypersurface in this case is a *symplectic mapping torus*.

To define the moment map of a torus action on a b-manifold, we first need to enlarge the codomain $\mathbb{R}^n$ to include points "at infinity." The preimage of these points will be the singular hypersurface of the b-symplectic manifold. We will also assign $\mathbb{R}$-valued weights to these points to encode certain geometric data, called the *modular periods* of the components of the singular hypersurface. The definition of a Delzant polytope generalizes in a natural way to this enlarged codomain, giving the definition of a *Delzant b-polytope*. The main theorem of this paper states that there is a bijection between b-symplectic toric manifolds and Delzant b-polytopes.

In contrast with classic symplectic geometry, the topology of the codomain of the moment map will depend on the b-manifold itself. This happens in two ways. First, to guarantee that the moment map is a smooth map, the smooth structure on the codomain will depend on the modular periods of the singular hypersurfaces of the b-manifold. Second, in some cases the codomain will not be contractible, but instead will be topologically a circle.

This Delzant theorem allows us to classify all $2n$-dimensional b-symplectic toric manifolds into two categories. The first kind of b-symplectic toric manifold has as its underlying manifold $X_\Delta \times \mathbb{T}^2$, where X_Δ is any classic $(2n-2)$-dimensional symplectic toric manifold. The second kind of b-symplectic toric manifold is constructed from the manifold $X_\Delta \times \mathbb{S}^2$ by a sequence of symplectic cuts, each performed along a hypersurface which does not intersect the singular hypersurface.

In the last section of this paper we consider possible extensions of our work. In particular, we discuss toric actions on b^2-manifolds (see [Sc]), toric actions on b-manifolds where the critical hypersurface may have normal crossings, and a cylindrical moment map for when the torus action on the b-manifold is not Hamiltonian but merely symplectic.

Last but not least, returning to the initial motivation of studying manifolds via their symmetries *à la Erlangen*, we study adapted integrable systems on b-manifolds and provide Eliasson-type theorem for integrable systems admitting non-degenerate singularities.

In a future paper, we plan to consider a refinement of the action-angle theorem for adapted integrable systems provided in [LMV] and an study of semitoric integrable systems on b-manifolds following the spirit of [PN2, PN1].

We also plan to consider general Hamiltonian actions of compact Lie groups and the study of the Marsden-Weinstein reduction procedure in this general framework will be considered in a future paper.

2. Preliminary definitions and examples

2.1. **b-objects, including b-functions.** We begin by recalling some of the notions introduced in detail in [GMP2]. A **b-manifold** is a pair (M, Z) consisting of an oriented smooth manifold M and a closed embedded hypersurface Z. A b-map $(M, Z) \to (M', Z')$ is an orientation-preserving map $f : M \to M'$ such that $f^{-1}(Z') = Z$ and f is transverse to Z'. We can define the **b-tangent bundle**, ${}^bTM$, whose sections are the vector fields on M which at points of Z are tangent to Z. The dual to this bundle is ${}^bT^*M$, the **b-cotangent bundle**. The smooth sections of $\Lambda^k({}^bT^*M)$ are called **b-de Rham k-forms** or simply **b-forms**. The space of all such forms is written ${}^b\Omega^k(M)$. The restriction of any b-de Rham k-form to $M\backslash Z$ is a classic differential k-form on $M\backslash Z$, and there is a differential $d : {}^b\Omega^k(M) \to {}^b\Omega^{k+1}(M)$ that extends the classic differential on $M\backslash Z$. With respect to this differential, we extend the standard definitions of closed and exact differential forms to **closed** b-forms and **exact** b-forms. A **b-symplectic** form is a closed b-form of degree 2 that has maximal rank (as a section of $\Lambda^k({}^bT^*M)$) at every point of M. A b-symplectic manifold consists of the data of a b-manifold (M, Z) together with a b-symplectic form ω. A **b-symplectomorphism** between two b-symplectic manifolds (M, ω) and (M', ω') is a b-map $\varphi : M \to M'$ such that $\varphi^*\omega' = \omega$.

Although a b-form can be thought of as a differential form with a singularity along Z, the singularity is so tame that it is even possible to define the integral of a form of top degree by taking its principal value near Z.

Definition 1. For any b-form $\eta \in {}^b\Omega^n(M)$ on a n-dimensional b-manifold and any local defining function y of Z, the **Liouville Volume** of η is

$$ {}^b\!\!\int_M \eta := \lim_{\varepsilon \to 0} \int_{M\backslash\{-\varepsilon \leq y \leq \varepsilon\}} \eta $$

The fact that the limit in Definition 1 exists and is independent of y is explained in [R] (for surfaces) and [Sc] (in the general case). Similarly, if $i_N : N \subseteq M$ is a k-dimensional submanifold transverse to Z, it inherits from

M a b-manifold structure $(N, i_N^{-1}(Z))$ and for any $\eta \in {}^b\Omega^k(M)$, we define

$$ {}^b\!\!\int_N \eta := {}^b\!\!\int_N i_N^* \eta. $$

In [GMP2], the authors prove that every b-form $\eta \in {}^b\Omega^p(M)$ can be written in a neighborhood of $Z = \{y = 0\}$ as

$$ \eta = \frac{dy}{y} \wedge \alpha + \beta $$

for smooth forms $\alpha \in \Omega^{p-1}(M)$ and $\beta \in \Omega^p(M)$. Although the forms α and β in this expression are not unique, the pullback $i_Z^*(\alpha)$ is unique, where i_Z is the inclusion $Z \subseteq M$. The resulting differential form on Z admits an alternative description: if v is a vector field on M such that $dy(v)\big|_Z = 1$, then the vector field $\mathbb{L} := yv$ is a b-vector field, $\mathbb{L}\big|_Z$ doesn't depend on v or y, the b-form $\iota_{\mathbb{L}}\eta$ is a smooth form, and $i_Z^*(\alpha) = i_Z^* \iota_{\mathbb{L}}\eta$. For this reason, we adopt the notation $\iota_{\mathbb{L}}\eta$ for this $(p-1)$-form on Z.

One can also study b-symplectic manifolds from the perspective of Poisson geometry: the dual of a b-symplectic form is a Poisson bivector whose top exterior product vanishes transversely (as a section of $\Lambda^{2n}(TM)$) at Z. Using these tools, we learn that Z has a codimension-one symplectic foliation. One important tool in the study of the geometry of this foliated hypersurface is the **modular vector field** on M. We review its definition.

Definition 2. Fix a volume form Ω on a b-symplectic manifold. The **modular vector field** $v_{\mathrm{mod}}^{\Omega}$ on M (or simply v_{mod} if Ω is clear from the context) is the vector field defined by the derivation

$$ f \mapsto \frac{\mathcal{L}_{u_f}\Omega}{\Omega}, $$

where u_f is the Hamiltonian vector field of the smooth function f on M defined by $df = \iota_{u_f}\omega$.

Although the modular vector field depends on Ω, different choices of Ω yield modular vector fields that differ by Hamiltonian vector fields. On a b-symplectic manifold, the modular vector field is tangent to the exceptional hypersurface Z and its flow preserves the symplectic foliation of Z, and Hamiltonian vector fields are tangent to the symplectic foliation.[1] In fact, in [GMP2] it is shown that corresponding to each modular vector field v_{mod} and compact leaf $\mathcal{L}$ of a component Z' of Z, there is a $k \in \mathbb{R}_{>0}$ and a symplectomorphism $f : \mathcal{L} \to \mathcal{L}$ such that Z' is the mapping torus

$$ \frac{\mathcal{L} \times [0, k]}{(\ell, 0) \sim (f(\ell), k)} $$

and the time-t flow of v_{mod} is translation by t in the second coordinate. The number k, which depends only on the choice of component $Z' \subseteq Z$, is called

[1] The reader should be aware that we will soon change our definition of "Hamiltonian vector fields" and this will no longer be true.

the **modular period** of Z'. This definition generalizes the one given in [R] for b-symplectic surfaces. Intuitively, the modular period of Z' is the time required for the modular vector field to flow a leaf of the foliation of Z' the entire way around the $\mathbb{S}^1$ base of the mapping torus.

Let $\mathcal{F}$ be the symplectic foliation induced by ω on Z, and for each symplectic leaf $\mathcal{L}$ let $i_{\mathcal{L}} : \mathcal{L} \hookrightarrow Z$ be the inclusion. A **defining one-form** for $\mathcal{F}$ (or more simply, for Z) is an $\alpha \in \Omega^1(Z)$ such that $\ker(\alpha_z) = T_z\mathcal{L} \subseteq T_z Z$ for each $z \in Z$. The authors of [GMP2] prove that $\iota_{\mathbb{L}}\omega$ is the unique defining one-form for Z that is both closed and satisfies $\alpha(v_{\text{mod}}) = 1$ for every modular vector field.

A **defining two-form** for Z is a non-vanishing $\beta \in \Omega^1(Z)$ such that $i_L^*\beta$ is the symplectic form induced by ω on the leaf L. We may always choose a defining two-form that is closed and satifies $\iota_{v_{\text{mod}}}\beta = 0$.

Not all closed b-forms on a b-manifold are locally exact. For example, if y is a local defining function for Z, then $\frac{dy}{y}$ is closed, but it is not exact in any neighborhood of any point of Z. Poincaré's lemma is such a fundamental property of the (smooth) de Rham complex that we are motivated to enlarge the sheaf C^∞ on a b-manifold to include functions such as $\log|y|$ so that we have a Poincaré lemma in b-geometry.

Definition 3. Let (M, Z) be a b-manifold. The sheaf [2] ${}^bC^\infty$ is defined by

$$ {}^bC^\infty(U) := \left\{ c\log|y| + f \;\middle|\; \begin{array}{l} c \in \mathbb{R} \\ y \text{ is any defining function for } U \cap Z \subseteq U \\ f \in C^\infty(U) \end{array} \right\} $$

Global sections of ${}^bC^\infty$ are called b**-functions**.

Replacing C^∞ with ${}^bC^\infty$ also enlarges the possible Hamiltonian torus actions on b-manifolds. For example, the action of $\frac{\partial}{\partial\theta}$ on $(\mathbb{S}^2, \{h = 0\}, \frac{dh}{h} \wedge d\theta)$ is generated by the Hamiltonian function $-\log|h| \in {}^bC^\infty(M)$, but is not generated by any function in $C^\infty(M)$. In fact, in Corollary 26 we show that there are no examples of effective Hamiltonian $\mathbb{T}^n$-actions on $2n$-dimensional b-symplectic manifolds with all their Hamiltonians in $C^\infty(M)$ except those with $Z = \emptyset$. We prove a simple relationship between the modular period and b-functions which will be useful in later sections.

Proposition 4. *Let (M, Z, ω) be a b-symplectic manifold and let Z' be a connected component of Z with modular period k. Let $\pi : Z' \to \mathbb{S}^1 \cong \mathbb{R}/k$ be the projection to the base of the corresponding mapping torus. Let $\gamma : \mathbb{S}^1 = \mathbb{R}/k \to Z'$ be any loop with the property that $\pi \circ \gamma$ is the positively-oriented loop of constant velocity 1. The following numbers are equal.*

- *The modular period of Z'.*

[2] Some authors require the hypersurface of a b-manifold to have a *global* defining function; other authors do not. If no global defining function for Z exists (for example, if Z is a meridian of $\mathbb{T}^2$), then this definition yields only a presheaf and ${}^bC^\infty$ is defined as its sheafification.

- $\int_\gamma \iota_{\mathbb{L}}\omega$.
- *The value of* $-c$ *for any* ${}^bC^\infty$ *function* $H = c\log|y| + f$ *in a neighborhood of* Z' *such that the corresponding Hamiltonian* X_H *has 1-periodic orbits homotopic in* Z' *to some* γ.

Proof. Recall from [GMP2] that $\iota_{\mathbb{L}}\omega(v_{\text{mod}})$ is the constant function 1. Let $s : [0,k] \to Z'$ be a trajectory of the modular vector field. Because the modular period is k, it follows that $s(0)$ and $s(k)$ are in the same leaf $\mathcal{L}$ of the foliation. Let $\hat{s} : [0,k+1] \to Z'$ be a smooth extension of s such that $s\big|_{[k,k+1]}$ is a path in $\mathcal{L}$ joining $\hat{s}(k) = s(k)$ to $\hat{s}(k+1) = s(0)$, making $\hat{s}$ a loop. Then

$$k = \int_0^k 1dt = \int_s \iota_{\mathbb{L}}\omega = \int_{\hat{s}} \iota_{\mathbb{L}}\omega = \int_\gamma \iota_{\mathbb{L}}\omega.$$

This shows that the first two numbers are equal.

Next, let $r : [0,1] \mapsto Z'$ be a trajectory of X_H, and notice that X_H satisfies $\iota_{X_H}\omega = c\frac{dy}{y} + df$. Let $y\frac{\partial}{\partial y}$ be a representative of $\mathbb{L}$. Because X_H is 1-periodic and homotopic to γ, it follows from the previous computation that

$$k = \int_r \iota_{\mathbb{L}}\omega = \int_0^1 \iota_{y\frac{\partial}{\partial y}}\omega(X_H\big|_{r(t)})dt = \int_0^1 -(c\frac{dy}{y} + df)(y\frac{\partial}{\partial y})\Big|_{r(t)} dt = -c$$

completing the proof. $\square$

2.2. Hamiltonian actions on symplectic and b-symplectic manifolds.

Let G be a compact connected Lie group which acts on a symplectic manifold M by symplectomorphisms, and denote by $\mathfrak{g}$ and $\mathfrak{g}^*$ its Lie algebra and corresponding dual, respectively. When $G = \mathbb{T}^n$, we write $\mathfrak{t}$ and $\mathfrak{t}^*$ instead of $\mathfrak{g}$ and $\mathfrak{g}^*$. We say that the action is **Hamiltonian** if there exists a map $\mu : M \to \mathfrak{g}^*$ which is equivariant with respect to the coadjoint action on $\mathfrak{g}^*$ such that for each element $X \in \mathfrak{g}$,

$$d\mu^X = \iota_{X^\#}\omega, \tag{1}$$

where $\mu^X =< \mu, X >$ is the component of μ in the direction of X, and $X^\#$ is the vector field on M generated by X:

$$X^\#(p) = \frac{d}{dt}\left[\exp(tX) \cdot p\right].$$

The map μ is called the **moment map**.

In the b-symplectic context, we restrict our attention to **b-symplectic torus actions**, that is, a torus acting by b-symplectomorphisms. We will notice that the definition of a *Hamiltonian* action and of a *moment map* must be adapted. To motivate the appropriate definitions we study two examples in detail.

Example 5. Consider the b-symplectic manifold $(\mathbb{S}^2, Z = \{h = 0\}, \omega = \frac{dh}{h} \wedge d\theta)$, where the coordinates on the sphere are the usual ones: $h \in [-1,1]$

and $\theta \in [0, 2\pi]$. For the usual $\mathbb{S}^1$-action given by the flow of $-\frac{\partial}{\partial\theta}$,

$$\iota_{-\frac{\partial}{\partial\theta}}\omega = \frac{dh}{h} = d(\log|h|),$$

so a moment map on $M \backslash Z$ is $\mu(h, \theta) = \log|h|$. The image of μ is drawn in Figure 1 as two superimposed half-lines depicted slightly apart to emphasize that each point in the image has two connected components in its preimage: one in the northern hemisphere, and one in the southern hemisphere. This phenomenon is dissimilar to classic symplectic geometry, where the level sets of Hamiltonians are connected and the moment map image of a symplectic toric manifold serves as a parameter space for the orbits of the $\mathbb{T}^n$-action. We also notice that the map μ is not defined on Z, even though the vector field from whence it came is defined on Z. In a later section, we will show that by interpreting the Hamiltonian as a section of ${}^bC^\infty$ and by enlarging the codomain of our moment map to include points "at infinity," we can define "moment maps" for torus actions on a b-manifold that enjoy many of the same properties as classic moment maps. In particular, they will be everywhere defined and their image will be a parameter space for the orbits of the action.

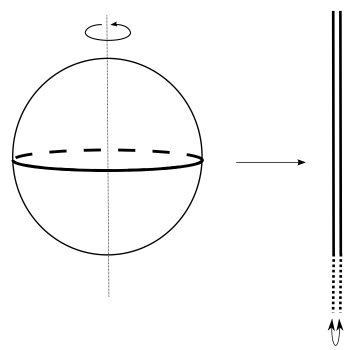


FIGURE 1. The moment map of $\mathbb{S}^1$ acting on a b-symplectic $\mathbb{S}^2$.

Example 6. Consider the b-symplectic manifold

$$(\mathbb{T}^2, Z = \{\theta_1 \in \{0, \pi\}\}, \omega = \frac{d\theta_1}{\sin\theta_1} \wedge d\theta_2)$$

where the coordinates on the torus are the usual ones: $\theta_1, \theta_2 \in [0, 2\pi]$. The exceptional hypersurface Z is the union of two disjoint circles. For the circle action of rotation on the θ_2 coordinate, because

$$\iota_{\frac{\partial}{\partial\theta_2}}\omega = -\frac{d\theta_1}{\sin\theta_1} = d\left(\log\left|\frac{1+\cos\theta_1}{\sin\theta_1}\right|\right),$$

the $\mathbb{S}^1$-action on $M \backslash Z$ is given by the ${}^bC^\infty$ Hamiltonian $\log\left|\frac{1+\cos\theta_1}{\sin\theta_1}\right|$.

The image of this function on $M \backslash Z$ is drawn in Figure 2. Each of the two connected components of $M \setminus Z$ is diffeomorphic to an open cylinder and maps to one of these lines. Again, notice that the preimage of a point in the image consists of two orbits.

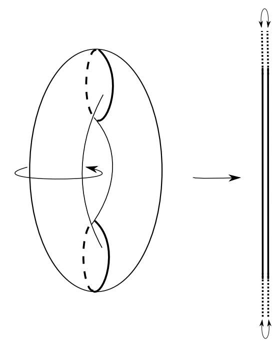


FIGURE 2. The moment map of $\mathbb{S}^1$ acting on a b-symplectic $\mathbb{T}^2$.

In both examples above, notice that although the Hamiltonian for the action on $M \backslash Z$ did not extend to a smooth function on all of M, it nevertheless extends to a ${}^bC^\infty$ function on all of M.

Definition 7. An action of $\mathbb{T}^n$ on a b-symplectic manifold (M, ω) is **Hamiltonian** if:

- for any $X \in \mathfrak{t}$, the one-form $\iota_{X^\#}\omega$ is exact, i.e., has a primitive $H_X \in {}^bC^\infty(M)$, and
- for any $X, Y \in \mathfrak{t}$, $\omega(X^\#, Y^\#) = 0$.

A Hamiltonian action is **toric** if it is effective and the dimension of the torus is half the dimension of M.

3. The b-Line and b-dual of the Lie algebra

When b-functions are the Hamiltonians of a torus action, we cannot expect to be able to gather them into a moment map $\mu : M \to \mathfrak{t}^*$ the same way we do in classic symplectic geometry: it would be impossible to define μ along Z. In this section, we define a moment map for a torus action on a b-manifold. To do so, we add points "at infinity" to the codomain $\mathfrak{t}^*$ to account for the singularities of b-functions. We begin our discussion with the simplest case: when the torus is simply a circle, we enlarge the line $\mathfrak{t}^* \cong \mathbb{R}$ into "the b-line" ${}^b\mathbb{R}$.

3.1. **The b-Line.** The b-line is constructed by gluing copies of the extended real line $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ together in a zig-zag pattern, then using $\mathbb{R}_{>0}$-valued labels ("weights") on the points at infinity to prescribe a smooth structure, and finally truncating the result to discard unneccessary copies of $\overline{\mathbb{R}}$. Figure 3 should help to put the technical details of the formal definition into a visual context.

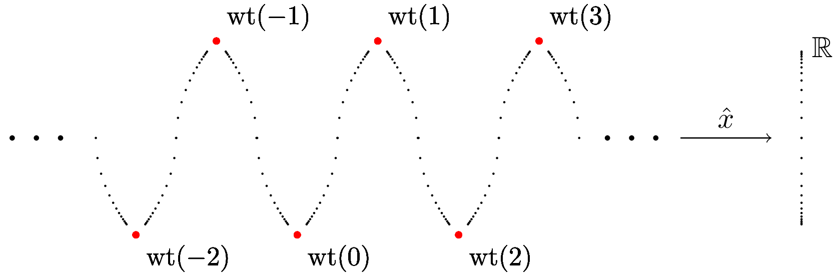


FIGURE 3. A weighted b-line with $I = \mathbb{Z}$.

Definition 8. Let $\mathrm{wt} : I \to \mathbb{R}_{>0}$, where I can be $\mathbb{Z}$ or $[1, N] \cap \mathbb{Z}$ or $[0, N] \cap \mathbb{Z}$. When $I = \mathbb{Z}$, the **b-line with weight function wt** is described as a topological space by

$$ {}_{\mathrm{wt}}^{\;b}\mathbb{R} \cong (\mathbb{Z} \times \overline{\mathbb{R}})/\{(a, (-1)^a\infty) \sim (a+1, (-1)^a\infty)\}. $$

Let $Z_{^b\mathbb{R}} = \mathbb{Z} \times \{\pm\infty\} \subseteq {}_{\mathrm{wt}}^{\;b}\mathbb{R}$, this set will function as an exceptional hypersurface of the manifold ${}_{\mathrm{wt}}^{\;b}\mathbb{R}$. Notice that ${}_{\mathrm{wt}}^{\;b}\mathbb{R}$ is homeomorphic to $\mathbb{R}$. The weight function prescribes a smooth structure[3] on ${}_{\mathrm{wt}}^{\;b}\mathbb{R}$ in the following way. Define

$$ \begin{aligned} \hat{x} : ({}_{\mathrm{wt}}^{\;b}\mathbb{R} \backslash Z_{^b\mathbb{R}}) = \mathbb{Z} \times \mathbb{R} &\to \mathbb{R} \\ (a, x) &\mapsto x \end{aligned} $$

and $\hat{y}_a : ((a-1, 0), (a, 0)) \to \mathbb{R}$ as

$$ \hat{y}_a = \begin{cases} -\exp\left((-1)^a \hat{x}/\mathrm{wt}(a)\right) & \text{on } ((a-1, 0), (a-1, (-1)^{a-1}\infty)) \\ 0 & \text{at } (a-1, (-1)^{a-1}\infty) \\ \exp\left((-1)^a \hat{x}/\mathrm{wt}(a)\right) & \text{on } ((a, (-1)^{a-1}\infty), (a, 0)) \end{cases}. $$

The coordinate maps $\{\hat{x}\big|_{\{a\}\times\mathbb{R}}, \hat{y}_a\}_{a\in\mathbb{Z}}$ define the structure of a smooth manifold on ${}_{\mathrm{wt}}^{\;b}\mathbb{R}$. When $I = [1, N] \cap \mathbb{Z}$ (respectively $[0, N] \cap \mathbb{Z}$), the weighted b-line ${}_{\mathrm{wt}}^{\;b}\mathbb{R}$ is defined as the open subset $((0, -\infty), (N, (-1)^N\infty))$ (respectively, $((-1, \infty), (N, (-1)^N\infty))$) of ${}_{\mathrm{wt}'}^{\;b}\mathbb{R}$, where $\mathrm{wt}' : \mathbb{Z} \to \mathbb{R}_{>0}$ is any function extending wt.

We will often abbreviate ${}_{\mathrm{wt}}^{\;b}\mathbb{R}$ by ${}^b\mathbb{R}$ when the weight function is understood from the context. To motivate the functions $\{\hat{y}_a\}$ in Definition 8, observe that

$$ \hat{x}\big|_{((a-1,0),(a,0))} = (-1)^a \mathrm{wt}(a) \log |\hat{y}_a| $$

This makes it possible to realize any $f \in {}^bC^\infty(M)$ on a b-manifold (M, Z) locally as a smooth map to a b-line.

[3]The reader may wonder why attention is being paid to define the smooth structure on ${}_{\mathrm{wt}}^{\;b}\mathbb{R}$ when a topological 1-manifold admits a unique smooth structure up to homeomorphism. The reason behind the care is because a homeomorphism intertwining two different smooth structures will not in general preserve the intrinsic affine structure present on each $\{a\} \times \mathbb{R} \subseteq_{\mathrm{wt}}^{\;b} \mathbb{R}$. This affine structure will be essential in the theory that follows.

Lemma 9. *Let (M, Z) be a b-manifold and Z' a connected component of Z. Any $f \in {}^bC^\infty(M)$ with a singularity at Z' can be expressed in a neighborhood of Z' as a smooth function F to a b-line ${}^b\mathbb{R}$. That is, $F^{-1}(Z_{{}^b\mathbb{R}}) = Z'$ and $\hat{x} \circ F = f$ on the complement of Z'.*

Proof. Let y be a local defining function for Z', and let U be a neighborhood of Z' on which $f\big|_U = c\log|y| + g$ for some $c \in \mathbb{R}, g \in C^\infty(U)$ and for which $U \backslash Z'$ has two connected components $\{U_+, U_-\}$. Because f is singular at Z', it follows that $c \neq 0$. If c is positive, let $\mathrm{wt} : \{0\} \mapsto c$ and define $F : U \to {}^b\mathbb{R}$ by the equation

$$\hat{y}_0 \circ F := \begin{cases} \exp(f/c) & \text{on } U_+ \\ -\exp(f/c) & \text{on } U_- \\ 0 & \text{on } Z \end{cases}$$

where the function $\hat{y}_0$ is defined in Definiton 8. If c is negative, let $\mathrm{wt} : \{1\} \mapsto -c$ and define $F : U \to {}^b\mathbb{R}$ by

$$\hat{y}_1 \circ F := \begin{cases} \exp(f/c) & \text{on } U_+ \\ -\exp(f/c) & \text{on } U_- \\ 0 & \text{on } Z \end{cases}$$

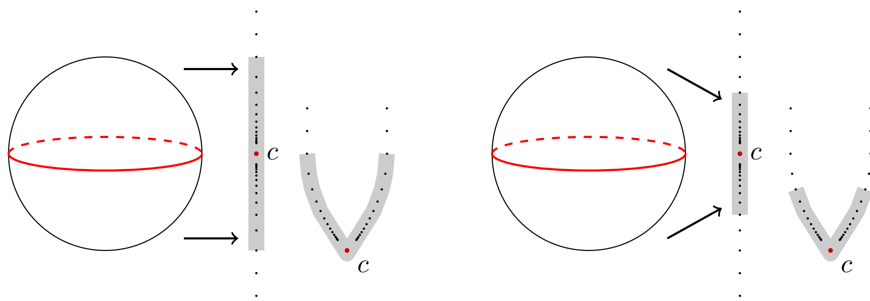
In both cases, the function F satisfies the conditions of the lemma. □

Remark 10. The function $\hat{y}_i \circ F$ constructed in the proof of Lemma 9 is a defining function for the hypersurface Z' that depends only on the original $f \in {}^bC^\infty(M)$ and the choice of which component of $U \backslash Z'$ to label U_+ and which to label U_-. Had we chosen this labelling differently, the resulting $\hat{y}_i \circ F$ would be replaced by its negative. Therefore, given a b-function f which is singular at Z', there is a canonical choice of defining function for Z' up to sign.

Remark 11. Not every b-function on every b-manifold can be *globally* expressed as a smooth function to a b-line. Consider when $M = \mathbb{S}^2$ and Z consists of two disjoint circles C_1 and C_2. Let y be a global defining function for Z, and pick a b-function on (M, Z) which restricts to $\log|y|$ and $2\log|y|$ in neighborhoods of C_1 and C_2 respectively. This b-function cannot be realized as a global map to any ${}^b\mathbb{R}$.

The following example illustrates Lemma 9 in the context of Hamiltonian torus actions.

Example 12. Let (h, θ) be the standard coordinates on $\mathbb{S}^2$. For any $c \in \mathbb{R}_{>0}$, the form $\omega_c = c\frac{dh}{h} \wedge d\theta$ is a b-symplectic form on $(\mathbb{S}^2, Z := \{h = 0\})$. Because $\iota_{-\frac{\partial}{\partial\theta}}\omega_c = c\frac{dh}{h}$, it follows that the b-function $c\log|h| + k$ for any $k \in \mathbb{R}$ is a Hamiltonian function generating the $\mathbb{S}^1$-action given by the flow of $-\frac{\partial}{\partial\theta}$. Figure 4 shows the map $\mu : \mathbb{S}^2 \to {}^b\mathbb{R}$ (with weight function $\mathrm{wt} : \{0\} \mapsto c$) corresponding to the Hamiltonian $c\log|h|$, and another μ' corresponding to $c\log|h| - 2$. In both cases, we have drawn ${}^b\mathbb{R}$ twice – the first is vertically so



$c \log |h| + 0$ as a map to ${}^b\mathbb{R}$ $\qquad$ $c \log |h| - 2$ as a map to ${}^b\mathbb{R}$

FIGURE 4. Two Hamiltonians generating the same $\mathbb{S}^1$-action.

that μ can be visualized as a projection, the second is bent so that it looks visually similar to the ${}^b\mathbb{R}$ in Figure 3.

There are two important observations to make about this example. The first is that the image of the moment maps $\mu = c \log |h|$ for different values of c have visually similar images – the only feature that distinguishes them is the numerical weight on the "point at infinity." This observation emphasizes the necessity of the weights: for different values of c, the b-manifolds (M, Z, ω_c) are *not* symplectomorphic. Were it not for the weight label, their moment map images would be indistinguishable. The second observation is that μ differs from μ' by changing the corresponding ${}^bC^\infty$ function by a constant. This shows that the picture of a "translation" of a b-line differs from the picture of a translation of $\mathbb{R}$.

Definition 13. Let ${}^b\mathbb{R}$ be a weighted b-line. A **translation** of ${}^b\mathbb{R}$ by $c \in \mathbb{R}$ is a map ${}^b\mathbb{R} \to {}^b\mathbb{R}$ which maps (a, b) to $(a, b + c)$ for finite values of b, and $(a, \pm\infty)$ to $(a, \pm\infty)$.

Using Definition 13, one can check that the images of μ and μ' shown in Figure 4 are translates of one another.

3.2. **b-dual of the Lie algebra.** Example 4 motivates the use of the b-line as a codomain for the moment map of a Hamiltonian $\mathbb{S}^1$-action on a b-surface. For a Hamiltonian $\mathbb{T}^n$-action on a symplectic b-manifold (M^{2n}, Z, ω) with $n > 1$, we will eventually prove that there always exists a subtorus $\mathbb{T}^{n-1}_Z \subseteq \mathbb{T}^n$ whose action is generated by vector fields tangent to the symplectic foliation of Z (even when Z is disconnected). The Lie algebra of this subtorus defines a hyperplane $\mathfrak{t}_Z$ in $\mathfrak{t}$ and dually a 1-dimensional subspace $(\mathfrak{t}_Z)^\perp$ in $\mathfrak{t}^*$. We will construct the codomain for the moment map of a toric action by replacing $(\mathfrak{t}_Z)^\perp \cong \mathbb{R}$ with a copy of ${}^b\mathbb{R}$, obtaining a space (non-canonically) isomorphic to ${}^b\mathbb{R} \times \mathbb{R}^{n-1}$.

Definition 14. Let $\mathfrak{t}$ be the Lie algebra of $\mathbb{T}^n$ and fix a primitive lattice vector $z \in \mathfrak{t}^*$ and a weight function $\mathrm{wt} : I \to \mathbb{R}_{>0}$ (again as in Definition 8, $I = \mathbb{Z}$ or $[0, N] \cap \mathbb{Z}$ or $[1, N] \cap \mathbb{Z}$). Write $\mathfrak{t}_Z$ for the hyperplane in $\mathfrak{t}$

perpendicular to z. When $I = \mathbb{Z}$, we define the **b-dual of the Lie algebra** ${}_{\mathrm{wt}}^{\,b}\mathfrak{t}^*$ (written ${}^b\mathfrak{t}^*$ when the weight function is clear from the context) to be the set

$${}_{\mathrm{wt}}^{\,b}\mathfrak{t}^* = (\mathbb{Z} \times \mathfrak{t}^*) \sqcup (\mathbb{Z} \times \mathfrak{t}_Z^*).$$

A choice of integral element $X \in \mathfrak{t}$ satisfying $\langle X, z\rangle = 1$ defines a set bijection

$$\begin{aligned} {}_{\mathrm{wt}}^{\,b}\mathfrak{t}^* = (\mathbb{Z} \times \mathfrak{t}^*) \sqcup (\mathbb{Z} \times \mathfrak{t}_Z^*) &\to {}_{\mathrm{wt}}^{\,b}\mathbb{R} \times \mathfrak{t}_Z^* \\ (a, \xi) &\longmapsto ((a, \langle \xi, X\rangle), [\xi]) \\ (a, [\xi]) &\longmapsto ((a, (-1)^{a+1}\infty), [\xi]) \end{aligned} \tag{2}$$

where the square brackets denote the image of an element of $\mathfrak{t}^*$ in $\frac{\mathfrak{t}^*}{\langle z\rangle} \cong \mathfrak{t}_Z^*$. The target space of the map (2) has a smooth b-manifold structure from Definition 8. This induces a smooth b-manifold structure on ${}_{\mathrm{wt}}^{\,b}\mathfrak{t}^*$. We will show in Proposition 15 that this structure is independent of the choice of X. When the domain of wt is a subset of $\mathbb{Z}$, we choose any $\mathrm{wt}' : \mathbb{Z} \to \mathbb{R}_{>0}$ that extends wt and define ${}_{\mathrm{wt}}^{\,b}\mathfrak{t}^*$ as the preimage (under the map (2)) of ${}_{\mathrm{wt}}^{\,b}\mathbb{R} \times \mathfrak{t}^* \subseteq {}_{\mathrm{wt}'}^{\,b}\mathbb{R} \times \mathfrak{t}^*$.

Proposition 15. *The smooth structure on ${}^b\mathfrak{t}^*$ is independent of the choice of X in its definition.*

Proof. Let X_1 and X_2 be integral elements of $\mathfrak{t}$ satisfying $\langle X_1, z\rangle = \langle X_2, z\rangle = 1$. This gives the following isomorphisms, where $\xi \in \mathfrak{t}^*$.

$$\begin{array}{ccccc} {}^b\mathbb{R} \times \mathfrak{t}_Z^* & \xleftarrow{\varphi_1} & (\mathbb{Z} \times \mathfrak{t}^*) \sqcup (\mathbb{Z} \times \mathfrak{t}_Z^*) & \xrightarrow{\varphi_2} & {}^b\mathbb{R} \times \mathfrak{t}_Z^* \\ ((a, \langle \xi, X_1\rangle), [\xi]) & \longleftarrow\!\shortmid & (a, \xi) & \longmapsto & ((a, \langle \xi, X_2\rangle), [\xi]) \\ ((a, (-1)^{a+1}\infty), [\xi]) & \longleftarrow\!\shortmid & (a, [\xi]) & \longmapsto & ((a, (-1)^{a+1}\infty), [\xi]) \end{array}$$

Because $X_2 - X_1 \in \mathfrak{t}_Z$, the map $\varphi_2 \circ \varphi_1^{-1}$ is given on $\mathbb{R} \times \mathfrak{t}_Z^*$ by

$$((a, x), [\xi]) \mapsto ((a, x + \langle [\xi], X_2 - X_1\rangle), [\xi]).$$

which is linear in the open coordinate charts $(\{a\} \times \mathbb{R}) \times \mathfrak{t}_Z^*$ of ${}^b\mathfrak{t}$. In the $\hat{y}_a$ coordinates, $\varphi_2 \circ \varphi_1^{-1}$ is given by

$$(\hat{y}_a, [\xi]) \mapsto (\hat{y}_a \exp((-1)^a \langle [\xi], X_2 - X_1\rangle / \mathrm{wt}(a)), [\xi])$$

which shows that the entire map $\varphi_2 \circ \varphi_1^{-1}$ is a diffeomorphism, proving that the smooth structures on ${}^b\mathfrak{t}^*$ induced by φ_1 and φ_2 are the same. $\square$

In practice, a Hamiltonian torus action on a b-manifold will not determine a natural choice of $z \in \mathfrak{t}^*$, but only the hypersurface $\mathfrak{t}_Z = z^\perp$. The reader may therefore find inelegant that the definition of ${}^b\mathfrak{t}^*$ depends on an arbitrary choice of z in Definition 14. However, a similar issue arises in classic symplectic geometry. Namely, given a Hamiltonian $\mathbb{S}^1$-action, the moment map $M \to \mathfrak{t}^*$ cannot be realized as a Hamiltonian function $M \to \mathbb{R}$ until an arbitrary choice has been made of which of the (two) lattice generators of $\mathfrak{t}^*$ to send to 1 in the identification $\mathfrak{t}^* \cong \mathbb{R}$. Choosing the opposite generator amounts to replacing the Hamiltonian function by its negative – in other words, postcomposing the Hamiltonian function with $\mathbb{R} \to \mathbb{R}$, $a \mapsto -a$. This situation is complicated in b-geometry by the sad fact that there is no automorphism φ of ${}^b\mathbb{R}$ that satisfies $\hat{x} \circ \varphi = -\hat{x}$. This can be seen from the fact that the b-line in Figure 3 does not have horizontal symmetry – you must follow your flip by a "horizontal shift" in order to realize an automorphism satisfying $\hat{x} \circ \varphi = -\hat{x}$. In other words, there is an automorphism of ${}^b_{\mathrm{wt}}\mathfrak{t}^*$ (using $z \in \mathfrak{t}^*$ as the distinguished lattice vector) and ${}^b_{\widetilde{\mathrm{wt}}}\mathfrak{t}^*$ (using $-z$ as the distinguished lattice vector), where $\widetilde{\mathrm{wt}}$ is defined by $\widetilde{\mathrm{wt}}(a) = \mathrm{wt}(a+1)$ (or $\widetilde{\mathrm{wt}}(a) = \mathrm{wt}(a-1)$, if the domain of wt is $[0, N]$). This is illustrated for the case when $\mathfrak{t}$ is 1-dimensional and wt has domain $[1, 3]$ in Figure 5. The

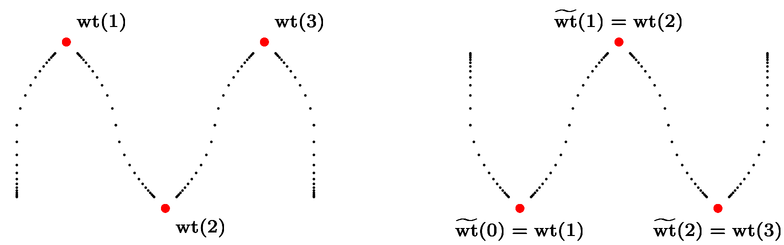


FIGURE 5. The effect of choosing a different distinguished direction.

reader who continues to find inelegant the choice of z in Definition 14 may prefer to write more general definitions of weight functions and of ${}^b\mathfrak{t}^*$ so that the two pictures in Figure 5 correspond to the same object.

Remark 16. Notice that for any $X \in \mathfrak{t}$ the map

$$ {}^b\mathfrak{t}^* \supseteq (\mathbb{Z} \times \mathfrak{t}^*) \to \mathbb{R}, \quad (a, \xi) \mapsto \langle \xi, X \rangle $$

extends to a b-function on ${}^b\mathfrak{t}^*$. This observation motivates the definition of a moment map.

Definition 17. Consider a Hamiltonian $\mathbb{T}^n$-action on a b-symplectic manifold (M, Z, ω), and let $\mu : M \to {}^b\mathfrak{t}^*$ be a smooth $\mathbb{T}^n$-invariant b-map. We say that μ is a **moment map** for the action if the map $X \mapsto \mu^X$ is linear and

$$ \iota_{X^\#}\omega = d\mu^X $$

where μ^X is the b-function $\mu^X(p) = \langle \mu(p), X \rangle$ described in Remark 16.

Example 18. Consider the b-symplectic manifold

$$ (M = \mathbb{S}^2 \times \mathbb{S}^2, Z = \{h_1 = 0\}, \omega = 3\frac{dh_1}{h_1} \wedge d\theta_1 + dh_2 \wedge d\theta_2) $$

where $(h_1, \theta_1, h_2, \theta_2)$ are the standard coordinates on $\mathbb{S}^2 \times \mathbb{S}^2$. The $\mathbb{T}^2$-action

$$ (t_1, t_2) \cdot (h_1, \theta_1, h_2, \theta_2) = (h_1, \theta_1 - t_1, h_2, \theta_2 - t_2) $$

is Hamiltonian. Let X_1 and X_2 be the elements of $\mathfrak{t}$ satisfying $X_1^\# = -\frac{\partial}{\partial \theta_1}$ and $X_2^\# = -\frac{\partial}{\partial \theta_2}$ respectively. Then $\mathfrak{t}_Z = \langle X_2 \rangle$. Letting $\mathrm{wt} : \{0\} \mapsto 3$ be the weight function, and $v = (X_1)^*$ be the distinguished direction in $\mathfrak{t}^*$, then we have a moment map $\mu : M \to {}^b\mathfrak{t}^*$ which can be described (using the basis $\{X_1, X_2\}$) as

$$M \to {}^b\mathbb{R} \times \mathbb{R}, \quad (h_1, \theta_1, h_2, \theta_2) \mapsto (\log |h_1|, h_2),$$

the image of which is illustrated in Figure 6.

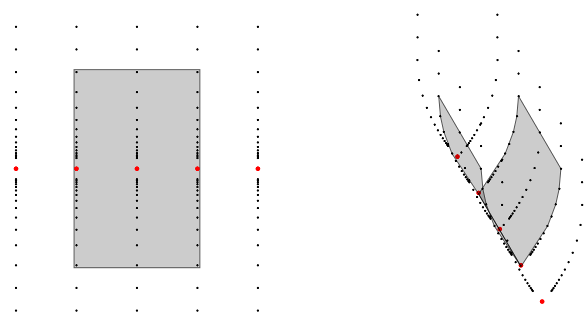


FIGURE 6. The moment map image $\mu(\mathbb{S}^2 \times \mathbb{S}^2)$, drawn twice.

The image on the left of Figure 6 shows the similarity between the moment map image and that of the standard action of $\mathbb{T}^2$ on $\mathbb{S}^2 \times \mathbb{S}^2$ from classic symplectic geometry. In the right image, the ${}^b\mathbb{R}$ factor of ${}^b\mathfrak{t}^*$ is bent to be visually similar to Figure 3.

In some cases, we must first quotient the codomain ${}^b\mathfrak{t}^*$ by a discrete group action to have a well-defined moment map.

Definition 19. Let $N \in \mathbb{Z}_{>0}$ be even and $\mathrm{wt} : [1, N] \to \mathbb{R}_{>0}$ be a weight function. Let $\mathrm{wt}' : \mathbb{Z} \to \mathbb{R}_{>0}$ be the N-periodic weight function that extends wt. Then ${}_{\mathrm{wt}}^{b}\mathbb{R}/\langle N \rangle$ (or just ${}^b\mathbb{R}/\langle N \rangle$) is defined as the quotient of ${}^b\mathbb{R}$ by the $\mathbb{Z}$ action $k \cdot (a, x) = (kN + a, x)$. Similarly, ${}_{\mathrm{wt}}^{b}\mathfrak{t}^*/\langle N \rangle$ is defined as the quotient of ${}_{\mathrm{wt}'}^{b}\mathfrak{t}^*$ by the smooth extension of the $\mathbb{Z}$ action $k \cdot (a, \xi) = (kN + a, \xi)$ on $\mathbb{Z} \times \mathfrak{t}^*$ to ${}_{\mathrm{wt}'}^{b}\mathfrak{t}^*$.

Topologically, the spaces ${}^b\mathbb{R}/\langle N \rangle$ and ${}^b\mathfrak{t}^*/\langle N \rangle$ are homeomorphic to a circle. The subset $Z_{{}^b\mathbb{R}}$ is preserved by the action described in Definition 19; its image in ${}^b\mathbb{R}/\langle N \rangle$ will be called $Z_{{}^b\mathbb{R}/\langle N \rangle}$. Similarly, the function $\hat{x}$ is well-defined on the complement of $Z_{{}^b\mathbb{R}/\langle N \rangle}$, and it still is the case that for any smooth b-map $\mu : M \to {}^b\mathfrak{t}^*$ and any $X \in \mathfrak{t}$, the function $p \mapsto \langle \mu(p), X \rangle$ on $M \backslash Z$ extends to a b-function on all of M. We define the notion of a moment map to the quotient spaces ${}^b\mathfrak{t}^*/\langle N \rangle$ in the same way as Definition 17.

Definition 20. Consider a Hamiltonian $\mathbb{T}^n$-action on a b-symplectic manifold (M, Z, ω), and let $\mu : M \to {}^b\mathfrak{t}^*/\langle N \rangle$ be a smooth $\mathbb{T}^n$-invariant b-map.

We say that μ is a **moment map** for the action if the map $X \mapsto \mu^X$ is linear and

$$\iota_{X^\#}\omega = d\mu^X$$

where μ^X is the b-function $\mu^X(p) = \langle \mu(p), X\rangle$.

Example 21. Consider the b-symplectic manifold

$$(\mathbb{T}^2 = \{(\theta_1, \theta_2) \in (\mathbb{R}/2\pi)^2\}, Z = \{\theta_1 \in \{0, \pi\}\}, \omega = \frac{d\theta_1}{\sin\theta_1} \wedge d\theta_2)$$

with $\mathbb{S}^1$-action given by the flow of $\frac{\partial}{\partial\theta_2}$. Let $X \in \mathfrak{t}$ be the element satisfying $X^\# = \frac{\partial}{\partial\theta_2}$. The weight function $\{0,1\} \mapsto 1$ and distinguished vector X^* define ${}^b\mathfrak{t}^*/\langle 2\rangle$, which we identify with ${}^b\mathbb{R}/\langle 2\rangle$ using the isomorphism induced by $X \in \mathfrak{t}$. A moment map for the $\mathbb{S}^1$-action is

$$\mu : \mathbb{T}^2 \to {}^b\mathbb{R}/\langle 2\rangle, \quad (\theta_1, \theta_2) \mapsto \begin{cases} (0, \infty) & \text{if } \theta_1 = 0 \\ \left(1, \log\left|\frac{1+\cos\theta_1}{\sin\theta_1}\right|\right) & \text{if } 0 < \theta_1 < \pi \\ (1, -\infty) & \text{if } \theta_1 = \pi \\ \left(0, \log\left|\frac{1+\cos\theta_1}{\sin\theta_1}\right|\right) & \text{if } \pi < \theta_1 < 2\pi \end{cases}$$

The reader is invited to check that this map is smooth. The image is shown in Figure 7.

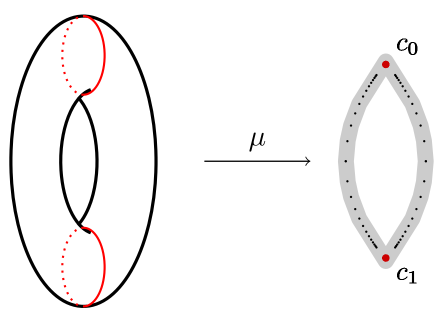


FIGURE 7. The moment map μ surjects onto ${}^b\mathfrak{t}^*/\langle 2\rangle$.

4. The moment map of a toric action on a b-symplectic manifold

4.1. **Local picture: in a neighborhood of Z.** Our first goal towards understanding toric actions on b-symplectic manifolds is to study their behavior near each connected component of Z. To simplify our exposition, *we assume throughout section 4.1 that Z is connected*, in the general case the local results hold in a neighborhood of each connected component of Z.

Proposition 30 is the main result of this section, it states that a toric action near (a connected component of) Z is locally a product of a codimension-1 torus action on a symplectic leaf of Z with an circle action whose flow is transverse to the leaves. This $\mathbb{T}^{n-1} \times \mathbb{S}^1$-action has a moment map whose image is the product of a Delzant polytope (corresponding to the action on the symplectic leaf) with an interval of ${}^b\mathbb{R}$.

The codimension-1 subtorus $\mathbb{T}^{n-1}$ will consist of those elements of $\mathbb{T}^n$ that preserve the symplectic foliation of Z. Toward the goal of showing that this subtorus is well-defined, we remind the reader of the following standard fact from Poisson geometry.

Remark 22. Let (M, Z, ω) be a b-symplectic manifold. Since Z is a Poisson submanifold of M, a Hamiltonian vector field X_f is tangent to the symplectic leaves of Z if and only if $f\big|_U \in C^\infty(U)$ for some neighborhood U of Z. In this case, if $i_{\mathcal{L}} : (\mathcal{L}, \omega_{\mathcal{L}}) \to Z$ is the inclusion of a symplectic leaf into Z, then $X_f\big|_{\mathcal{L}} = X_{f\circ i_{\mathcal{L}}}$.

Given a Hamiltonian $\mathbb{T}^k$-action on (M^{2n}, Z, ω) and any $X \in \mathfrak{t}$, the b-one-form $\iota_{X^\#}\omega$ has a ${}^bC^\infty$ primitive that can be written in a neighborhood of Z as $c \log|y| + g$, where y is a local defining function for Z, the function g is smooth, and $c \in \mathbb{R}$ depends on X. The map $X \mapsto c$ is an element v_Z of $\mathfrak{t}^* = \mathrm{Hom}(\mathfrak{t}, \mathbb{R})$ (we invite the reader to verify that c does not depend on the choices involved and that $X \mapsto c$ is a homomorphism). We will denote by $\mathfrak{t}_Z$ the kernel of v_Z. By Proposition 4, the values of $\langle v_Z, X\rangle$ are integer multiples of the modular period of Z when X is a lattice vector. Therefore, we conclude that v_Z is rational. We will show in Claim 25 that v_Z is nonzero. First, we prove an equivariant Darboux theorem for compact group actions in a neighborhood of a fixed point. Given a fixed point p of an action $\rho : G \times M \longrightarrow M$, we denote by $d\rho$ the linear action defined via the exponential map in a neighborhood of the origin in T_pM: $d\rho(g, v) = d_p(\rho(g))(v)$.

Theorem 23. *Let ρ be a b-symplectic action of a compact Lie group G on the b-symplectic manifold (M, Z, ω), and let $p \in Z$ be a fixed point of the action. Then there exist local coordinates $(x_1, y_1, \ldots, x_{n-1}, y_{n-1}, z, t)$ centered at p such that the action is linear in these coordinates and*

$$\omega = \sum_{i=1}^{n-1} dx_i \wedge dy_i + \frac{1}{z}\, dz \wedge dt.$$

Proof. After choosing a metric near p, the exponential map gives a diffeomorphism ϕ from a neighborhood U of $0 \in T_pM$ to a neighborhood of $p \in M$. By choosing the metric wisely we can guarantee that $\phi(U \cap T_pZ) \subseteq Z$. Pulling back under ϕ the group action and symplectic form on M to a group action and symplectic form on T_pM, it suffices to prove the theorem for the b-manifold (T_pM, T_pZ). Therefore, assume that ω and ρ live on (T_pM, T_pZ).

By Bochner's theorem [B], the action of ρ is locally equivalent to the action of $d\rho$. That is, there are coordinates $(x_1, y_1, \ldots, x_{n-1}, y_{n-1}, z, t)$ centered at $\mathbf{0} = (0, 0, \ldots, 0)$ on which the action is linear. By studying the construction of φ in [B], we see that the coordinates can be chosen so that T_pZ is the coordinate hyperplane $\{z = 0\}$. Also, after a linear change of these coordinates, we may assume that

$$\omega\big|_{\mathbf{0}} = \sum_{i=1}^{n-1} dx_i \wedge dy_i + \frac{1}{z}\, dz \wedge dt.$$

Next, we will perform an equivariant Moser's trick. Let $\omega_0 = \omega$,

$$\omega_1 = \sum_{i=1}^{n-1} dx_i \wedge dy_i + \frac{1}{z}\, dz \wedge dt, \quad \text{and} \quad \omega_s = s\omega_1 + (1-s)\omega_0, \text{ for } s \in [0,1].$$

Because ω_s has full rank at $\mathbf{0}$ for all s, we may assume (after shrinking the neighborhood) that ω_s has full rank for all s. Let α be a primitive for $\omega_1 - \omega_0$ that vanishes at $\mathbf{0}$ (α is a b-form), and let X_s be the b-vector field defined by the equation

$$\iota_{X_s}\omega_s = -\alpha.$$

Since X_s is a b-vector field that vanishes at $\mathbf{0}$, its flow preserves Z and fixes $\mathbf{0}$. The time one flow of X_s is a symplectomorphism $(T_pM, \omega_0) \to (T_pM, \omega_1)$, but this symplectomorphism will not in general be equivariant, and so there is no guarantee that the action is still linear. We therefore pick a Haar measure μ on G and consider the vector field

$$X_s^G = \int_G \rho(g)_*(X_s)d\mu.$$

The vector field X_s^G commutes with the group action. Since $\rho(g)$ preserves ω_0 and ω_1, it also preserves ω_s for all s. Therefore, the averaged vector field satisfies the equation

$$\iota_{X_s^G}\omega_s = -\int_G \rho(g)^*(\alpha)d\mu.$$

Observe also that the new invariant b-one-form $\alpha_G = \int_G \rho(g)^*(\alpha)d\mu$ is also a primitive for $\omega_1 - \omega_0$ due to $d\rho$-invariance of the family of b-forms ω_s. Thus, the flow of X_s^G commutes with the linear action and satisfies the equation

$$\iota_{X_s^G}\omega_s = -\alpha_G.$$

Therefore the time one flow of X_s^G takes ω_0 to ω_1 in an equivariant way. $\square$

In the particular case where the group is a torus we obtain the following:

Corollary 24. *Consider a fixed point $z \in Z$ of a symplectic $\mathbb{T}^k$-action on (M, Z, ω). If the isotropy representation on T_zM is trivial, then the action is trivial in a neighborhood of z.*

Claim 25. Let (M^{2n}, Z, ω) be a b-symplectic manifold with a toric action, and assume that Z is connected. Then v_Z is nonzero. As a consequence, $\mathfrak{t}_Z$ is a hyperplane in $\mathfrak{t}$.

Proof. Consider a toric action on (M^{2n}, Z, ω) with the property that $\iota_{X^\#}\omega \in \Omega^1(M)$ for every $X \in \mathfrak{t}$. It suffices to prove that such an action is not effective. Let $(\mathcal{L}, \omega_{\mathcal{L}})$ be a leaf of the symplectic foliation of Z. By Remark 22 the action on M induces a toric action on the symplectic manifold $(\mathcal{L}, \omega_{\mathcal{L}})$. Because $\dim(\mathcal{L}) = 2n-2$, there must be a subgroup $\mathbb{S}^1 \subseteq \mathbb{T}^n$ that acts trivially on $\mathcal{L}$.

For any $z \in \mathcal{L}$, the isotropy representation of this $\mathbb{S}^1$-action on T_zM restricts to the identity on $T_z\mathcal{L} \subseteq T_zM$ and preserves the subspace T_zZ. It therefore induces a linear $\mathbb{S}^1$-action on the 1-dimensional vector space $T_zZ/T_z\mathcal{L}$. Any such action is trivial, so it follows that the isotropy representation restricts to the identity on T_zZ. Following the same argument, the isotropy representation on all of T_zM is trivial. By Corollary 24, this shows that the $\mathbb{S}^1$-action is the identity on a neighborhood of z, so the action is not effective. □

In the general case, there will be a different element $v_{Z'}$ for each connected component Z' of Z, but we will see in Claims 34 and 35 that they will be nonzero scalar multiples of one another and therefore that the corresponding hyperplanes $\mathfrak{t}_{Z'}$ are all the same.

Corollary 26. *If the b-symplectic manifold (M, Z, ω) admits a toric action with the property that each $\iota_{X^\#}\omega \in \Omega^1(M)$ for every $X \in \mathfrak{t}$, then $Z = \emptyset$.*

Proposition 27. *Let (M^{2n}, Z, ω) be a b-symplectic manifold with a toric action. Let X be a representative of a primitive lattice vector of $\mathfrak{t}/\mathfrak{t}_Z$ that pairs positively with v_Z. Then $\langle X, v_Z\rangle$ equals the modular period of Z.*

Proof. By Proposition 4, it suffices to prove that a time-1 trajectory of $X^\#$ that starts on Z, when projected to the $\mathbb{S}^1$ base of the mapping torus Z, travels around the loop just once. Let $p \in \mathbb{R}^+$ be the smallest number such that $\Phi_p^X(\mathcal{L}) = \mathcal{L}$, where $\Phi_p^{X^\#}$ is the time-p flow of $X^\#$. The condition that $\omega(X^\#, Y^\#) = 0$ for all $Y \in \mathfrak{t}_Z$ implies that the symplectomorphism $\Phi_p^{X^\#}\Big|_{\mathcal{L}}$ preserves the $\mathbb{T}_Z$-orbits of $\mathcal{L}$. We can realize any such symplectomorphism as the time-1 flow of a Hamiltonian vector field v on the symplectic leaf $(\mathcal{L}, \omega_{\mathcal{L}})$ (see, for example, the proof of Proposition 6.4 in [LT]). The product of the $\mathbb{T}_Z$ action with the flow of $p^{-1}v$ defines a Hamiltonian $\mathbb{T}_Z \times \mathbb{S}^1 \cong \mathbb{T}^n$ action on the $(\mathcal{L}, \omega_{\mathcal{L}})$, so there exists $\mathbb{S}^1 \subseteq \mathbb{T}_Z \times \mathbb{S}^1$ that acts trivially on $\mathcal{L}$. Since the $\mathbb{T}_Z$ action is effective, this $\mathbb{S}^1$ is not a subset of $\mathbb{T}_Z$. Therefore we may assume, after replacing X with $X+Y$ for some $Y \in \mathfrak{t}_Z$, that the time-p flow of $X^\#$ is the identity on $\mathcal{L}$. Then, for any $z \in \mathcal{L}$, the isotropy representation of the time-p flow of $X^\#$ would be the identity on T_zM, proving (by Corollary 24) that the time-p flow of $X^\#$ is the identity in a neighborhood of z. By effectiveness, $p = 1$. □

In particular, Proposition 27 proves that the trajectories of $X^\#$ inside Z, when projected to the $\mathbb{S}^1$ base of the mapping torus Z, travel around the loop just once. Because $X^{\#}$ is periodic and preserves the symplectic foliation, the flow of $X^{\#}$ defines a product structure on Z.

Corollary 28. *Let (M^{2n}, Z, ω) be a b-symplectic manifold with a toric action and assume that Z is connected. Let $\mathcal{L}$ be a symplectic leaf of Z. Then,*

$$Z \cong \mathcal{L} \times \mathbb{S}^1.$$

In the general case, this result implies that each connected component Z' is of the form $\mathcal{L}' \times \mathbb{S}^1$, for possibly distinct $\mathcal{L}'$. We will see however that the existence of a global toric action forces all $\mathcal{L}'$ to be identical.

We are nearly ready to prove Proposition 30, which states that locally in a neighborhood of Z, a toric action splits as the product of a $\mathbb{T}^{n-1}_Z$-action and an $\mathbb{S}^1$-action. We preface its proof by studying a related example in classic symplectic geometry – the intuition gained from this informal discussion will prepare the reader for the proofs of Lemma 29 and Proposition 30.

Consider the symplectic manifold $M = \mathbb{S}^2 \times \mathbb{S}^2, \omega = dh_1 \wedge d\theta_1 + dh_2 \wedge d\theta_2$ with a Hamiltonian $\mathbb{T}^2$-action defined by

$$(t_1, t_2) \cdot (h_1, \theta_1, h_2, \theta_2) = (h_1, \theta_1 + t_1, h_2, \theta_2 + t_2).$$

Let $\{X_1, X_2\}$ be the basis of $\mathfrak{t}$ such that $X_1^{\#} = \frac{\partial}{\partial \theta_1}$ and $X_2^{\#} = \frac{\partial}{\partial \theta_2}$. After identifying $\mathfrak{t}^*$ with $\mathbb{R}^2$ using the dual basis, the $\mathbb{T}^2$-action is given by the moment map (h_1, h_2) with image $\Delta = [-1, 1]^2$.

Consider the two hypersurfaces $Z_1 = \{h_2 = 0\}, Z_2 = \{h_1 + h_2 = -1\}$ in M as shown in Figure 8. Near L_1, Δ is locally the product $L_1 \times (-\varepsilon, \varepsilon)$; near

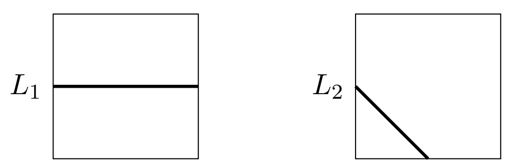


FIGURE 8. Hypersurfaces in $\mathbb{S}^2 \times \mathbb{S}^2$: $Z_1 = \mu^{-1}(L_1)$ and $Z_2 = \mu^{-1}(L_2)$.

L_2, Δ is not locally a product. The vector field $u = \frac{\partial}{\partial h_1}$ in a neighborhood of Z_1 has the property that $dh_1(u) = 1$, and $\omega(Y^{\#}, u) = 0$ for all Y in the hypersurface of $\mathfrak{t}$ spanned by X_2. If we flow Z_1 along the vector field u, the image under μ would consist of the line segment L_1 moving with constant velocity in the direction perpendicular to $\langle X_2 \rangle$, showing once again that Δ is locally the product $L_1 \times (-\varepsilon, \varepsilon)$ near L_1. In contrast, there is no vector field u' in a neighborhood of Z_2 such that $d(h_1+h_2)(u') = 1$ and $\omega(Y^{\#}, u') = 0$ for all Y in a hypersurface of $\mathfrak{t}$, reflecting the fact that Δ is not locally a product near L_2. The reason that no such u' exists is because every hypersurface of $\mathfrak{t}$ contains some Y such that $\iota_{Y^{\#}}\omega$ is a multiple of $d(h_1 + h_2)$ somewhere along Z_2 (making the condition that $d(h_1 + h_2)(u') = 1$ incompatible with $\omega(Y^{\#}, u') = 0$). In other words, the fact that Δ is locally a product near

L_1 is reflected in the fact that $\ker(\iota_{Y^\#}\omega_z) \neq T_zZ$ for all $z \in Z$ and all Y in some hyperplane of $\mathfrak{t}_Z$.

In a neighborhood of the exceptional hypersurface Z of a b-manifold, a toric action will always behave similarly to Z_1, in the sense that the hyperplane $\mathfrak{t}_Z \subseteq \mathfrak{t}$ satisfies the property $\ker(\iota_{Y^\#}\omega_z) \neq T_zZ$ for all $z \in Z$ and $Y \in \mathfrak{t}_Z$. This fact is the content of Lemma 29 and will play an important role in the proof of Proposition 30.

Lemma 29. *Let $k < n$ and consider a Hamiltonian $\mathbb{T}^k$-action on (M^{2n}, Z, ω) for which $\iota_{X^\#}\omega \in \Omega^1(M)$ for each $X \in \mathfrak{t}$. Then for any $z \in Z$ and $X \in \mathfrak{t}$, $\ker(\iota_{X^\#}\omega_z) \neq T_zZ$.*

Proof. Let u be vector field defined in a neighborhood of Z with the property that u is transverse to Z and is $\mathbb{T}^k$-invariant (for example, by picking any transverse vector field and averaging). Let Φ_t^u be the time-t flow along u. For sufficiently small ε

$$\phi : Z \times (-\varepsilon, \varepsilon) \to U, \quad (z,t) \mapsto \Phi_t^u(z)$$

is a diffeomorphism onto a neighborhood U of Z. Let p and y be the projections of $Z \times (-\varepsilon, \varepsilon)$ onto Z and $(-\varepsilon, \varepsilon)$ respectively. Then

$$\phi^*(\omega) = \frac{dt}{t} \wedge p^*(\alpha) + \beta$$

where $\alpha \in \Omega^1(Z)$ is given by $\iota_{\mathbb{L}}(\omega)$ and β is a smooth 2-form on $Z \times (-\varepsilon, \varepsilon)$.

Let $V \subseteq Z$ be a neighborhood of $z \in Z$ for which $\alpha\big|_V$ has a primitive $\theta' \in C^\infty(V)$, and define $\theta := p^*(\theta')$. Pick functions $\{x_i\}$ such that $\{t, \theta, x_1, \ldots, x_{2n-2}\}$ are coordinates in a neighborhood of $(z,0) \in Z \times (-\varepsilon, \varepsilon)$. Then we can write $X^\#$ and $\phi^*\omega$ in these coordinates

$$\begin{aligned} X^\# &= v_\theta \frac{\partial}{\partial \theta} + v_t \frac{\partial}{\partial t} + \sum_i v_i \frac{\partial}{\partial x_i} \\ \phi^*\omega &= \frac{dt}{t} \wedge d\theta + w_{t\theta} dt \wedge d\theta + \sum_i (w_{ti} dt \wedge dx_i + w_{\theta i} d\theta \wedge dx_i) \\ &\qquad + \sum_{ij} w_{ij} dx_i \wedge dx_j \end{aligned}$$

where the subscripted v's and w's are smooth functions. Because the kernel of the covector $\iota_{X^\#}\omega_z$ has dimension either $2n$ or $2n-1$, it is enough to show that if $z \in Z$ and $X \in \mathfrak{t}$ are such that $\ker(\iota_{X^\#}\omega_z) \supseteq T_zZ$, then actually $\iota_{X^\#}\omega_z = 0$, which happens exactly if the dt term of $\iota_{X^\#}\omega$ vanishes at z. The coefficient of the dt term of $\iota_{X^\#}\omega$ is

$$-\left(\frac{v_\theta}{t} + v_\theta w_{t\theta} + \sum_i v_i w_{ti}\right) \tag{3}$$

Because u was chosen to be $\mathbb{T}^k$-invariant, $\frac{\partial}{\partial t}$ is also $\mathbb{T}^k$-invariant, so

$$0 = \left[\frac{\partial}{\partial t}, X^\#\right](\theta) = \frac{\partial}{\partial t}\left(X^\#(\theta)\right) - X^\#\left(\frac{\partial}{\partial t}(\theta)\right) = \frac{\partial}{\partial t}\left(d\theta(X^\#)\right) = \frac{\partial}{\partial t}(v_\theta)$$

This shows that $\frac{\partial v_\theta}{\partial t}$ also vanishes at z. Because $X^\#$ is the Hamiltonian vector field of a smooth function H_X, it is tangent to the symplectic leaf at z and can be calculated (by Remark 22) as the Hamiltonian vector field (using the symplectic form on the leaf) of the pullback of H_X to the leaf. Since we are assuming that $\ker(\iota_{X^\#}\omega_z)$ contains T_zZ (and in particular $T_z\mathcal{L}$), it follows that the pullback of H_X has a critical point at z. Therefore, $(X^\#)_z = 0$, which proves that v_θ and all v_i vanish at z. This shows that the term (3) also vanishes at z, proving the claim. □

Proposition 30. *Let (M^{2n}, Z, ω) be a b-symplectic manifold with a toric action and assume that Z is connected. Let c be the modular period of Z and $\mathcal{L}$ a leaf of its symplectic foliation. Pick a lattice element $X_b \in \mathfrak{t}$ that represents a generator of $\mathfrak{t}/\mathfrak{t}_Z$ and pairs positively with v_Z.*

Then there is a neighborhood $\mathcal{L} \times \mathbb{S}^1 \times (-\varepsilon, \varepsilon) \cong U \subseteq M$ of Z such that the $\mathbb{T}^n$-action has moment map

$$\mu : \mathcal{L} \times \mathbb{S}^1 \times (-\varepsilon, \varepsilon) \to {}^b\mathfrak{t}^* \cong {}^b\mathbb{R} \times \mathfrak{t}_Z^*, \qquad (\ell, \rho, t) \mapsto (y_0 = t, \mu_{\mathcal{L}}(\ell)) \tag{4}$$

where the weight function on ${}^b\mathbb{R}$ is given by $\{0\} \mapsto c$, the map $\mu_{\mathcal{L}} : \mathcal{L} \to \mathfrak{t}_Z^*$ is a moment map for the $\mathbb{T}_Z^{n-1}$-action on $\mathcal{L}$, and the isomorphism ${}^b\mathfrak{t}^* \cong {}^b\mathbb{R} \times \mathfrak{t}_Z^*$ is the one described in Definition 14 using X_b as the primitive lattice element.

Proof. Observe that the splitting $\mathfrak{t} \cong \langle X_b \rangle \oplus \mathfrak{t}_Z$ induces a splitting $\mathbb{T}^n \cong \mathbb{S}^1 \times \mathbb{T}^{n-1}$. Pick a primitive f_b of $\iota_{X_b}\omega$. Let $y_Z : U \to M$ be a defining function for Z corresponding to f_b (as defined in Remark 10) in some neighborhood U of Z. Because f_b is $\mathbb{T}^n$-invariant, so too is y_Z, since the level sets of y_Z coincide with those of f_b. Our first goal is to pick a vector field u in a neighborhood of Z' with the following three properties.

(1) $dy_Z(u) = 1$
(2) $\iota_{Y^\#}\omega(u) = 0$ for all $Y \in \mathfrak{t}_Z$
(3) u is $\mathbb{T}^n$-invariant

To show that a vector field exists that satisfies conditions (1) and (2) simultaneously, it suffices to observe that for each $z \in Z$ and $Y \in \mathfrak{t}_Z$, $\ker(\iota_{Y^\#}\omega_z) \neq T_zZ$ by Lemma 29. Let u be a vector field satisfying (1) and (2). Because dy_Z and each $\iota_{Y^\#}\omega$ are $\mathbb{T}^n$-invariant, we can average u by the $\mathbb{T}^n$-action without disturbing properties (1) and (2). Therefore, by replacing u with its $\mathbb{T}^n$-average we may assume that u is $\mathbb{T}^n$-invariant. Let Φ_t^u and $\Phi_t^{X_b^\#}$ be the time-t flows of u and $X_b^\#$ respectively. Then, using Corollary 28, the map

$$\phi : \mathcal{L} \times \mathbb{S}^1 \times (-\varepsilon, \varepsilon) \to U, \quad (\ell, \rho, t) \mapsto \Phi_t^u \circ \Phi_\rho^{X_b^\#}(\ell)$$

is a diffeomorphism for sufficiently small ε. Let p and t be the projections of $\mathcal{L} \times \mathbb{S}^1 \times (-\varepsilon, \varepsilon)$ onto $Z \cong \mathcal{L} \times \mathbb{S}^1$ and $(-\varepsilon, \varepsilon)$ respectively. To study the induced $\mathbb{T}^n$-action on the domain of ϕ, fix some $(s, g) = (\exp(kX_b), \exp(Y)) \in \mathbb{S}^1 \times \mathbb{T}^{n-1}$ and recall that since u is $\mathbb{T}^n$-invariant, its flows commute with the flows of all $\{X^\# \mid X \in \mathfrak{t}\}$. If we denote the $\mathbb{T}^{n-1}$-action on $\mathcal{L}$ by $g \cdot_{\mathcal{L}} \ell$, then

$$\begin{aligned}\phi(g \cdot_{\mathcal{L}} \ell, \rho + s, t) &= \Phi_t^u \circ \Phi_{\rho+s}^{X_b^\#}(g \cdot_{\mathcal{L}} \ell) = \Phi_t^u \circ \Phi_\rho^{X_b^\#} \circ \Phi_s^{X_b^\#} \circ \Phi_1^{Y^\#}(\ell) \\ &= \Phi_s^{X_b^\#} \circ \Phi_1^{Y^\#} \phi(\ell, \rho, t) = (s, g) \cdot \phi(\ell, \rho, t)\end{aligned}$$

which shows that the induced $\mathbb{T}^n$-action on the domain is given by

$$(s, g) \cdot (\ell, \rho, t) = (g \cdot_{\mathcal{L}} \ell, \rho + s, t).$$

We will show that the moment map for this is given by (4). Notice that $\mu^{X_b} \in {}^bC^\infty(\mathcal{L} \times \mathbb{S}^1 \times (-\varepsilon, \varepsilon))$ is given by $c \log|t|$, and $X_b^\#$ is $\frac{\partial}{\partial \rho}$. Then

$$\iota_{X_b^\#} \phi^* \omega = \phi^*(\iota_{X_b^\#} \omega) = \phi^*(df_b) = d\phi^*(c \log|y_Z|) = (-1)^a c \frac{dt}{t} = d\mu^{X_b}$$

as required. To prove that $\iota_{Y^\#}(\phi^*\omega) = d\mu^Y$ for $Y \in \mathfrak{t}_Z$, first we define the map

$$p_{\mathcal{L}} : U \to \mathcal{L}, \quad \phi(\ell, \rho, t) \mapsto \ell$$

and observe that $p_{\mathcal{L}} \circ \phi(\ell, \rho, t) = \ell$. Also, since the map $p_{\mathcal{L}}$ can be realized at $\phi(\ell, \rho, t)$ as the time-$(-t)$ flow of u followed by the time-$(-\rho)$ flow of $X_b^\#$, both of which preserve $\iota_{Y^\#}\omega$, it follows that $p_{\mathcal{L}}^*(\iota_{Y^\#}\omega) = \iota_{Y^\#}\omega$. Then

$$\iota_{Y^\#}(\phi^*\omega) = \phi^*(\iota_{Y^\#}\omega) = \phi^* p_{\mathcal{L}}^*(\iota_{Y^\#}\omega) = (p_{\mathcal{L}} \circ \phi)^*(d\mu_{\mathcal{L}}^Y) = d\mu^Y$$

where the final equality follows from the fact that $p_{\mathcal{L}} \circ \phi(\ell, \rho, t) = \ell$. $\square$

Notice that if we had chosen X_b to be a generator of $\mathfrak{t}/\mathfrak{t}_Z$ that pairs negatively with v_Z, then by the discussion following Proposition 15, the moment map for the action would have y_1 appearing in the place of y_0, and the weight function $\{1\} \mapsto c$ instead of $\{0\} \mapsto c$.

Because the torus action preserves each component of Z and each component of $M \backslash Z$, it induces an action on the open symplectic manifold W, the restriction of a moment map $\mu : M \to {}^b\mathfrak{t}^*$ to a single connected component W of $M \backslash Z$ gives a moment map (in a classic sense) by identifying each $\{a\} \times \mathfrak{t}^* \subseteq {}^b\mathfrak{t}^*$ with $\mathfrak{t}^*$:

$$\mu_W : W \to \mathfrak{t}^*$$

Restricting the moment map described in Proposition 30 in this way gives the following result.

Corollary 31. *Let (M^{2n}, Z, ω) be a b-symplectic manifold with a toric action and assume that Z is connected. Let W be a connected component of*

$M\backslash Z$. Then there is a neighborhood $U \subseteq M$ of Z such that the $\mathbb{T}^n$-action on $U \cap W$ has moment map with image the Minkowski sum[4]

$$\Delta + \{kv_Z \mid k \in \mathbb{R}^-\}$$

where $\Delta \subseteq \mathfrak{t}^*$ is an affinely embedded copy of the image of $\mu_{\mathcal{L}} : \mathcal{L} \to \mathfrak{t}_Z^*$ into $\mathfrak{t}^*$.

The next proposition describes a local model for the b-symplectic manifold in a neighborhood of Z. It will be necessary in the proof of the Delzant theorem to show that the moment map is unique.

Proposition 32. *(Local Model) Let (M^{2n}, Z, ω) be a b-symplectic manifold with a toric action and assume Z is connected. Fix ${}^b\mathfrak{t}^*$ with $wt(1) = c$ and some $X \in \mathfrak{t}$ representing a lattice generator of $\mathfrak{t}/\mathfrak{t}_Z$ that pairs positively with the distinguished vector v, inducing an isomorphism ${}^b\mathfrak{t}^* \cong {}^b\mathbb{R} \times \mathfrak{t}_Z^*$. For any Delzant polytope $\Delta \subseteq \mathfrak{t}_Z^*$ with corresponding symplectic toric manifold $(X_\Delta, \omega_\Delta, \mu_\Delta)$, define the **local model** b-symplectic manifold as*

$$M_{lm} = X_\Delta \times \mathbb{S}^1 \times \mathbb{R} \qquad \omega_{lm} = \omega_\Delta + c\frac{dt}{t} \wedge d\theta$$

where θ and t are the coordinates on $\mathbb{S}^1$ and $\mathbb{R}$ respectively. The $\mathbb{S}^1 \times \mathbb{T}_Z$ action on M_{lm} given by $(\rho, g) \cdot (x, \theta, t) = (g \cdot x, \theta + \rho, t)$ has moment map $\mu_{lm}(x, \theta, t) = (y_0 = t, \mu_\Delta(x))$.

For any toric action on a b-manifold (M, Z, ω) with moment map μ such that $\mu(U) = (-\epsilon \le y_0 \le \epsilon) \times \Delta$ in a neighborhood U of Z, then there is an equivariant b-symplectomorphism $\varphi : M_{lm} \to M$ in a neighborhood of $X_\Delta \times \mathbb{S}^1 \times \{0\}$ satisfying $\mu \circ \varphi = \mu_{lm}$.

Proof. Fix a symplectic leaf $\mathcal{L} \subseteq Z$. Because μ maps Z surjectively to $\{y_0 = 0\} \times \Delta$ and μ is $\mathbb{T}^n$-invariant, it must be the case that $\mathrm{im}(\mu\big|_{\mathcal{L}}) = \{y_0 = 0\} \times \Delta$. Define $\mu_{\mathcal{L}} : \mathcal{L} \to \mathfrak{t}_Z^*$ to be the projection of $\mu\big|_{\mathcal{L}}$ onto its second coordinate. By the classic Delzant theorem there is an equivariant symplectomorphism $\varphi_\Delta : (X_\Delta, \omega_\Delta) \to (\mathcal{L}, \omega_{\mathcal{L}})$ such that $\mu_\Delta = \mu_{\mathcal{L}} \circ \varphi_\Delta$. As in the proof of Proposition 30, let y_Z be a local defining function for Z corresponding to a primitive of $\iota_X\omega$ and let u be a $\mathbb{T}^n$-equivariant vector field in a neighborhood of Z, such that $dy_Z(u) = 1$ and $\iota_{Y^\#}\omega(u) = 0$ for all $Y \in \mathfrak{t}_Z$. Then the map

$$\varphi : M_{\mathrm{lm}} = X_\Delta \times \mathbb{S}^1 \times \mathbb{R} \to M, \quad (x, \theta, t) \mapsto \Phi_t^u \circ \Phi_\theta^{X^\#} \circ \varphi_\Delta(x)$$

is defined in a neighborhood of $X_\Delta \times \mathbb{S}^1 \times \{0\}$.

It follows by the equivariance of $u, X^\#$, and φ_Δ that φ itself is equivariant. Next, observe that

$$\mu \circ \varphi(x, \theta, t) = \mu \circ \Phi_\theta^{X^\#} \circ \Phi_t^u \circ \varphi_\Delta(x) = \mu \circ \Phi_t^u \circ \varphi_\Delta(x)$$

since μ is $\mathbb{T}^n$-invariant. Observe that the $\mathfrak{t}_Z^*$-component of $\mu \circ \Phi_t^u \circ \varphi_\Delta(x)$ will equal $\varphi_\Delta(x)$, since $\iota_{Y^\#}\omega(u) = 0$ for all $Y \in \mathfrak{t}_Z$. The ${}^b\mathbb{R}$-component of

[4] The Minkowski sum of two sets A and B is $A + B = \{a + b \mid a \in A, b \in B\}$.

$\mu \circ \Phi_t^u \circ \varphi_\Delta(x)$ will equal t, since the $X^\#$ action is generated by the b-function $(y_0 = y_Z)$ and the vector field u satisfies $dy_Z(u) = 1$. Therefore, $\mu \circ \varphi = \mu_{\mathrm{lm}}$.

This shows that on the b-symplectic manifolds $(M_{\mathrm{lm}}, \omega_{\mathrm{lm}})$ and $(M_{\mathrm{lm}}, \varphi^*(\omega))$, the same moment map μ_{lm} corresponds to the same action. Our next goal is to show that $\varphi^*\omega\big|_Z = \omega_{\mathrm{lm}}\big|_Z$. For $z \in Z$, let $A \subseteq {}^bT_zM$ be the symplectic orthogonal to $(X^\#)_z$. Restriction of the canonical map ${}^bT_zM \to T_zM$ to A leaves its image unchanged (since the kernel of the canonical map, $\mathbb{L}$, is not in A). By picking a basis for $T_zL \subseteq T_zZ$ and pulling it back to A, and then adding $(X^\#)_z$ and $(t\frac{\partial}{\partial t})_z$, we obtain a basis of ${}^bT_zZ$. By calculating the value of ω_z with respect to this basis, and using the facts that φ_Δ is a symplectomorphism and that

$$\varphi^*\omega(t\frac{\partial}{\partial t}, \frac{\partial}{\partial \theta}) = \omega(y_Z u, X^\#) = d(c \log |y_Z|)(y_Z u) = c,$$

we conclude that $\varphi^*\omega\big|_Z = \omega_{\mathrm{lm}}\big|_Z$. To complete the proof, we will carefully apply Moser's path method to construct a symplectomorphism between $\varphi^*\omega$ and ω_{lm}.

Note that $\varphi^*\omega - \omega_{\mathrm{lm}}$ is $\mathbb{T}^n$-invariant and has the property that the tangent space to each orbit is in the kernel of $\varphi^*\omega - \omega_{\mathrm{lm}}$. Therefore, we can write $\varphi^*\omega - \omega_{\mathrm{lm}}$ as the pullback under μ_{lm} of a smooth form ν on ${}^b\mathbb{R} \times \mathfrak{t}_Z^*$. Let α be the pullback (under μ_{lm}) of a primitive of ν. Then α is a primitive of $\varphi^*\omega - \omega_{\mathrm{lm}}$ with the property that the vector fields defined using Moser's path method will be tangent to the orbits of the torus action, and also with the property that α is torus invariant. Therefore, the equivariant symplectomorphism it defines leaves the moment map unchanged, completing the proof.

□

4.2. **Global picture.** Let (M^{2n}, Z, ω) be a b-symplectic manifold with a toric action. As before, for a connected component W of $M \setminus Z$, we write $\mu_W : W \to \mathfrak{t}^*$ for the moment map on W induced by μ.

Claim 33. The image $\mu_W(W)$ is convex.

Proof. Let $Z_1, \dots, Z_r$ be the connected components of Z which are in the closure of W. By Proposition 30, we can find a function t_i in a neighborhood of Z_i for which an $\mathbb{S}^1$ factor of the $\mathbb{T}^n$-action is generated by the Hamiltonian $c \log |t_i|$ for some $c \neq 0$. Define $W_{\geq\varepsilon} \subseteq W$ to be $W \backslash \{|t_i| < \varepsilon\}$, let $W_{=\varepsilon}$ be its boundary, and let $W_{>\varepsilon} = W_{\geq\varepsilon} \backslash W_{=\varepsilon}$. Figure 9 shows W with $W_{>\varepsilon}$ shaded.[5]

Performing a symplectic cut at $W_{=\varepsilon}$ gives a compact symplectic toric manifold $\overline{W_{\geq\varepsilon}}$ which has an open subset canonically identified with $W_{>\varepsilon}$. Let $\mu_{W,\varepsilon} : \overline{W_{\geq\varepsilon}} \to \mathfrak{t}^*$ be the moment map for the toric action on $\overline{W_{\geq\varepsilon}}$ that agrees with μ_W on $W_{>\varepsilon}$. To show that $\mu_W(W)$ is convex, pick points

[5] With Claims 34 and 35 we will see that the number of connected components of Z adjacent to W can be at most two. We have drawn three connected components of Z adjacent to W in Figure 9 so that the figure is more pedagogically effective at the expense of accuracy.

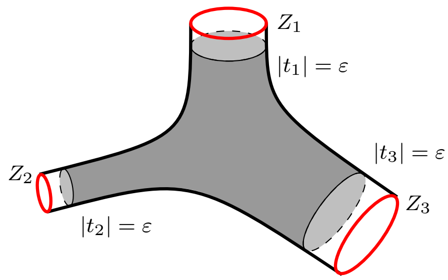


FIGURE 9. A connected component W of $M\backslash Z$ and the open subset $W_{>\varepsilon}$.

$\mu_W(p), \mu_W(q)$ in $\mu_W(W)$ and fix some $\varepsilon > 0$ small enough that that $p, q \in W_{>\varepsilon}$. Because $\overline{W_{\geq\varepsilon}}$ is compact, $\mu_{W,\varepsilon}(\overline{W_{\geq\varepsilon}})$ contains the straight line joining $\mu_W(p) = \mu_{W,\varepsilon}(p)$ and $\mu_W(q) = \mu_{W,\varepsilon}(q)$. Since $\mu_{W,\varepsilon}(\overline{W_{\geq\varepsilon}}) \subseteq \mu_W(W)$, the image $\mu_W(W)$ also contains the straight line joining $\mu_W(p)$ and $\mu_W(q)$. □

By Corollary 31, we know that for each connected component Z' of Z adjacent to W, there is a neighborhood U of Z' such that $\mu_W(U_i \cap W)$ is the product of a Delzant polytope with the ray generated by $-v_{Z'}$. By performing symplectic cuts near the hypersurfaces adjacent to W (as in the proof of Claim 33) to partition the image of μ_W into a convex set and these infinite prisms, we see that the convex set $\mu_W(W)$ extends indefinitely in precisely the directions

$$\{-v_{Z'}\}_{Z' \text{ is adjacent to } W}. \tag{5}$$

Claim 34. Each of these directions occupy the same one-dimensional subspace of $\mathfrak{t}^*$. That is, $\mathfrak{t}_{Z'}$ is independent of the choice of component $Z' \subseteq Z$.

Proof. Pick $x_1, x_2 \in \mu_W(W)$ such that the rays

$$\{x_1 + tv_1 \mid t \in \mathbb{R}_{>0}\} \qquad \text{and} \qquad \{x_2 + tv_2 \mid t \in \mathbb{R}_{>0}\}$$

are both in $\mu_W(W)$ (for example, by taking x_1 and x_2 to be images of points in the neighborhoods of Z_1 and Z_2 described in in Corollary 31). By convexity of $\mu_W(W)$ the point y_t below is in $\mu_W(W)$ for all $t \geq 0$ and any $\lambda \in [0, 1]$.

$$y_t = \lambda(x_1 + tv_1) + (1-\lambda)(x_2 + tv_2) = \lambda x_1 + (1-\lambda)x_2 + t(\lambda v_1 + (1-\lambda)v_2)$$

which proves that there is a ray in $\mu_W(W)$ that extends infinitely far in the $(\lambda v_1 + (1-\lambda)v_2)$ direction. Because there are only finitely many directions in which $\mu_W(W)$ extends indefinitely far, x_1 must be a scalar multiple of x_2. □

As a consequence of Claim 34 and of convexity, we must have that $\mu_W(W)$ extends indefinitely in one direction or in two opposite directions. The next claim shows that each of these "infinite directions" corresponds to only one connected component of Z.

Claim 35. Suppose that Z_1 and Z_2 are two different connected components of Z both adjacent to the same connected component W of $M \backslash Z$. Then $v_{Z_1} = k v_{Z_2}$ for some $k < 0$.

Proof. By Claim 34, $v_{Z_1} = k v_{Z_2}$ for some $k \in \mathbb{R}$, and by Claim 25, $k \neq 0$. It suffices, therefore, to prove that k cannot be positive. Assume towards a contradiction that k is positive, and pick a lattice element $X \in \mathfrak{t}$ such that $\langle X, v_{Z_1} \rangle = 1$, and let $H : W \to \mathbb{R}$ be a Hamiltonian for the $\mathbb{S}^1$-action generated by X. By performing symplectic cuts sufficiently close to the components of Z adjacent to W (as in the proof of Claim 33) and using the fact that the level sets of moment maps on compact connected symplectic manifolds are connected, it follows that the level set $H^{-1}(\lambda)$ is connected for any $\lambda \in \mathbb{R}$. In a neighborhood of Z_1 and of Z_2, the function H approaches negative infinity. Therefore, for sufficiently large values of N, the level set $H^{-1}(-N)$ has a connected component completely contained in a neighborhood of Z_1 and another connected component completely contained in a neighborhood of Z_2. Because $H^{-1}(-N)$ has just one connected component, $Z_1 = Z_2$. □

In particular, this means that in M, each component of $M \backslash Z$ is adjacent to at most two connected components of Z.

Definition 36. The **adjacency graph** G_M of a symplectic b-manifold (M, Z, ω) is a graph with a vertex for each component of $M \backslash Z$ and an edge for each connected component of Z that connects the vertices corresponding to the components of $M \backslash Z$ that it separates.

When (M^{2n}, Z, ω) has an effective toric action, this graph must either a loop or a line, as illustrated in Figure 10. If it is a loop, Claim 35 implies that it must have an even number of vertices.

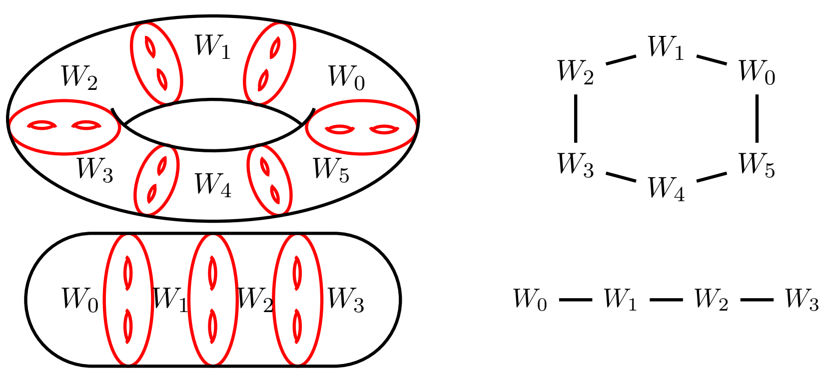


FIGURE 10. The adjacency graph is either a cycle of even length or a line.

We are finally ready to prove the main theorem of of this section: that every b-symplectic manifold with a toric action has a moment map.

Theorem 37. *Let $(M, Z, \omega, \mathbb{T}^n)$ be a b-symplectic manifold with an effective Hamiltonian toric action. For an appropriately-chosen ${}^b\mathfrak{t}^*$ or ${}^b\mathfrak{t}^*/\langle N\rangle$, there is a moment map $\mu : M \to {}^b\mathfrak{t}^*$ or $\mu : M \to {}^b\mathfrak{t}^*/\langle N\rangle$.*

Proof. Consider the adjacency graph of the connected components of $M\backslash Z$ as described in Figure 10. We first consider the case when the graph is a line. Number the components $W_0, \ldots, W_{N-1}$ as described, and let Z_i be the connected hypersurface between W_{i-1} and W_i. Let c_i be the modular period of Z_i. Then consider the ${}^b\mathfrak{t}^*$ defined using weight function $\mathrm{wt}(i) = c_i$ and the primitive lattice vector in the direction of $-v_{Z_1}$. Fix any moment map $\mu_{W_0} : W_0 \to \mathfrak{t}^*$ for the action on W_0. By identifying the codomain $\mathfrak{t}^*$ of this moment map with $\{0\} \times \mathfrak{t}^* \subseteq {}^b\mathfrak{t}^*$, we get a moment map $\mu_{W_0} : W_0 \to {}^b\mathfrak{t}^*$. By Proposition 30, there is a moment map μ_{U_1} for the $\mathbb{T}^n$-action in a neighborhood U_1 of Z_1. The two moment maps

$$\mu_{W_0}\big|_{W_0\cap U_1} \quad \text{and} \quad \mu_{U_1}\big|_{W_0\cap U_1}$$

correspond to the same $\mathbb{T}^n$-action on $W_0 \cap U_1$, so by postcomposing μ_{U_1} with a translation we may glue μ_{W_0} and μ_{U_1} into a moment map defined on all of $W_0 \cup U_1$. We continue extending the moment map in this manner until it is a moment map μ defined on all of M. As a consequence of this construction, notice that μ maps the component W_i into $\{i\} \times \mathfrak{t}^* \subseteq {}^b\mathfrak{t}^*$; this motivates the decision to label the components of $M\backslash Z$ starting with 0 instead of 1.

When the adjacency graph is a cycle, consider performing the above construction using the weight function defined on $\mathbb{Z}$ which is N-periodic with $\mathrm{wt}(i) = c_i$ for $0 \le i \le N-1$. The construction breaks down in the final stage; after choosing the correct translation of the moment map μ_{U_N} so that it agrees with $\mu_{W_{N-1}}$ on the overlap of their domains, it will not be the case that μ_{U_N} agrees with μ_{W_0} on the overlap of their domains. Pick some $p \in U_N \cap W_0$, and define

$$x_{\text{start}} = \mu_{W_0}(p) \quad \text{and} \quad x_{\text{end}} = \mu_{U_N}(p)$$

and assume without loss of generality that $x_{\text{start}} = (0, 0) \in \mathbb{Z} \times \mathfrak{t}^* \subseteq {}^b\mathfrak{t}^*$. Let $\gamma : \mathbb{S}^1 = \mathbb{R}/\mathbb{Z} \to M$ be a loop with $\gamma(0) = \gamma(1) = p$ that visits the sets $W_0, U_1, W_1, \ldots, W_{N-1}, U_N$ in that order. Then, for any $X \in \mathfrak{t}$, we have

$$x_{\text{end}} = (N, x) \quad \text{where } \mu^X(x) = \begin{cases} \int_\gamma \iota_{X^\#}\omega & X \in \mathfrak{t}_Z \\ {}^b\!\!\int_\gamma \iota_{X^\#}\omega & X \notin \mathfrak{t}_Z \end{cases}$$

When $X \in \mathfrak{t}_Z$, the 1-form $\iota_{X^\#}\omega$ has a smooth primitive, so this integral equals zero. When $X \notin \mathfrak{t}_Z$, the 1-form $\iota_{X^\#}\omega$ does not have a smooth primitive, but still has a ${}^bC^\infty$ primitive, and the Liouville volume of the pullback is still zero. And therefore the integral equals zero. Therefore, $x_{\text{end}} = (N, 0)$ and the moment maps for each the sets W_i and U_i glue into a moment map $\mu : M \to {}^b\mathfrak{t}^*/\langle N\rangle$. $\square$

Theorem 37 proves that every effective Hamiltonian toric action has a moment map. However, as in the classic case (where different translations of the moment map correspond to the same action), the moment map is not uniquely determined by the action. To better understand the variety of moment maps that can correspond to a torus action on a b-manifold, we review the arbitrary choices made during the construction of the moment map. Clearly, the adjacency graph as well as the modular periods are determined uniquely by the b-symplectic manifold, but the labelling of the vertices is not. When the graph is a line, we chose which leaf of the vertex to label W_0 and which to label W_{N-1}; when the graph is a cycle, we chose which vertex to label W_0 and in which direction around the cycle the graph should increase (or when $N = 2$, which lattice generator of $\mathfrak{t}^*_Z$ to distinguish in the construction of ${}^b\mathfrak{t}^*$). As such, the moment map is unique not only up to translation, but also up to certain permutations of the domain of the weight function and possibly a different choice of distinguished element of $\mathfrak{t}^*_Z$. The effect of changing the distinguished lattice vector necessitates a notational juggling which is described in Proposition 15 and illustrated in Figure 5. For the upcoming Delzant theorem, we will incorporate into our definition of a *b-symplectic toric manifold* the data of a moment map. Not only does this follow the precedent of the classic definition of a symplectic toric manifold, but it also relieves from us the burden of following and notating these choices made in constructing a moment map.

5. Delzant theorem

In this section, we prove a Delzant theorem in b-geometry. Towards this goal, we define the notion of a b-symplectic toric manifold, and a Delzant b-polytope.

Definition 38. A **b-symplectic toric manifold** is

$$(M^{2n}, Z, \omega, \mu : M \to {}^b\mathfrak{t}^*) \quad \text{or} \quad (M^{2n}, Z, \omega, \mu : M \to {}^b\mathfrak{t}^*/\langle a \rangle)$$

where (M, Z, ω) is a b-symplectic manifold and μ is a moment map for a toric action on (M, Z, ω).

Notice that the definition of a b-symplectic toric manifold also implicitly includes the information of a weight function and a distinguished lattice vector used to construct ${}^b\mathfrak{t}^*$. As in the classic case, the definition of a polytope in ${}^b\mathfrak{t}^*$ will use the definition of a half-space in ${}^b\mathfrak{t}^*$. The definition of a half-space is an intuitive concept obfuscated by notation; we encourage the reader to look at the examples in Figure 11 before reading the formal definition. Although the boundaries of the half-spaces in Figure 11 appear curved, they are actually straight lines when restricted to each $\{k\} \times \mathfrak{t}^* \cong \mathfrak{t}^*$; they appear curved only because of the way ${}^b\mathfrak{t}^*$ is drawn. Notice that the boundary of a half-space will not intersect $Z_{{}^b\mathfrak{t}^*}$ unless it is perpendicular to it.

Definition 39. For a fixed ${}^b\mathfrak{t}^*$ with weight function with domain $[a, N]$ for $a \in \{0, 1\}$ and distinguished vector $v \in \mathfrak{t}^*$, consider the two following kinds

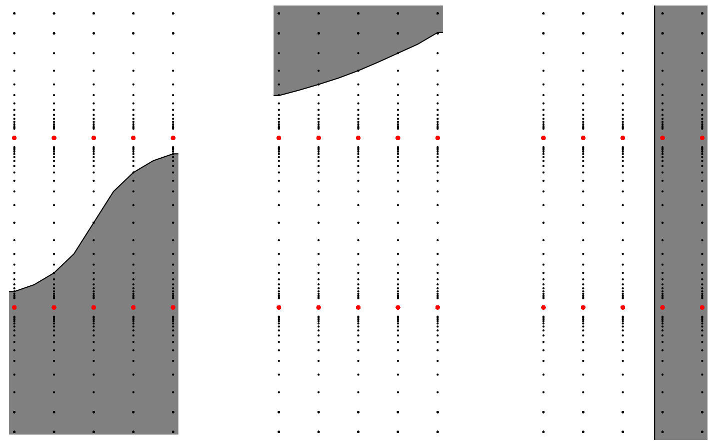


FIGURE 11. Examples of half-spaces in ${}^b\mathfrak{t}^*$.

of hypersurfaces in ${}^b\mathfrak{t}^*$, where $X \in \mathfrak{t}$, $Y \in v^\perp$, $k \in \mathbb{R}$ and $c \in [a-1, N]$:

$$A_{X,k,c} = \{(c,\xi) \mid \langle \xi, X\rangle = k\} \subseteq \{c\} \times \mathfrak{t}^* \subseteq {}^b\mathfrak{t}^*,$$
$$B_{Y,k} = \overline{\{(c,\xi) \mid \langle \xi, Y\rangle = k, c \in [a-1,N]\}} \subseteq \overline{[a-1,N] \times \mathfrak{t}^*} = {}^b\mathfrak{t}^*.$$

The complement of any such hypersurface is two connected components in ${}^b\mathfrak{t}^*$. The closure of any such connected component is a **half-space** in ${}^b\mathfrak{t}^*$. The same definitions of $A_{X,k,c}$ and $B_{X,k}$ also define hypersurfaces in ${}^b\mathfrak{t}^*/\langle N\rangle$ when N is even. But in this case, the hypersurfaces of type $A_{X,k,c}$ do not separate the space. Therefore, only the closure of a connected component of the complement of some $B_{X,k} \subseteq {}^b\mathfrak{t}^*/\langle N\rangle$ is called a **half-space** in ${}^b\mathfrak{t}^*/\langle N\rangle$.

In Figure 11, the first two images are examples of a half-space corresponding to some $A_{X,k,c}$, while the rightmost image is an example of a half-space corresponding to $B_{X,k}$.

Definition 40. A b**-polytope** in ${}^b\mathfrak{t}^*$ (or ${}^b\mathfrak{t}^*/\langle N\rangle$) is a bounded subset P that intersects each component of $Z_{{}^b\mathfrak{t}^*}$ (or $Z_{{}^b\mathfrak{t}^*/\langle N\rangle}$) and can be expressed as a finite intersection of half-spaces.

If the condition that P must intersect each component of $Z_{{}^b\mathfrak{t}^*}$ were removed from the definition of a polytope, then for any pair of weight functions $\mathrm{wt}' : [a, N'] \to \mathbb{R}_{>0}, \mathrm{wt} : [a, N] \to \mathbb{R}_{>0}$ such that wt' extends wt, any polytope in ${}_{\mathrm{wt}}^{\ b}\mathfrak{t}^*$ would also be a polytope in ${}_{\mathrm{wt}'}^{\ b}\mathfrak{t}^*$ under the inclusion ${}_{\mathrm{wt}}^{\ b}\mathfrak{t}^* \subseteq {}_{\mathrm{wt}'}^{\ b}\mathfrak{t}^*$. The upcoming statement of Theorem 45, which generalizes the Delzant theorem, is easier to state when we disallow this non-uniqueness of the weight function.

Example 41. Figure 12 shows two examples of b-polytopes. In both cases, the torus has dimension two. The polytope on the left is a subset of ${}^b\mathfrak{t}^* \cong {}^b\mathbb{R} \times \mathbb{R}$, and the polytope on the right is a subset of ${}^b\mathfrak{t}^*/\langle 2\rangle$ (the top of the picture on the right is identified with the bottom of the picture).

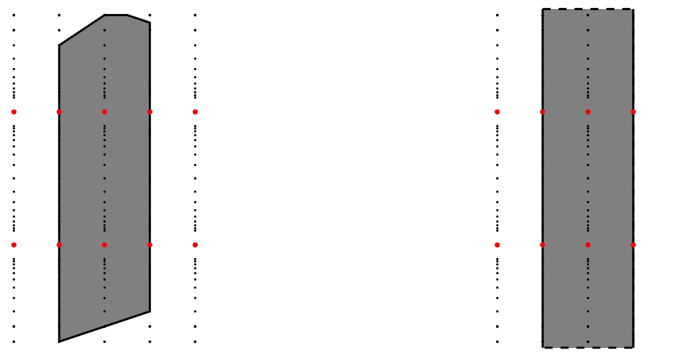


FIGURE 12. Examples of a polytope in ${}^b\mathfrak{t}^*$ and one in ${}^b\mathfrak{t}^*/\langle N\rangle$.

The definitions of many features of classic polytopes, such as **facets**, **edges**, and **vertices**, generalize in a natural way to b-polytopes, as does the notion of a **rational** polytope (one in which the X's and k's in Definition 39 are rational). We state some properties of b-polytopes, all of which are straightforward consequences of the definition.

- The hypersurface $A_{X,k,c}$ separates $Z_{{}^b\mathfrak{t}^*} = [a,N]\times\mathfrak{t}_Z^*$ into $[a,c-1]\times\mathfrak{t}_Z^*$ and $[c,N]\times\mathfrak{t}_Z^*$. Because of the condition that P must intersect each component of $Z_{{}^b\mathfrak{t}^*}$, the only hypersurfaces of type $A_{X,k,c}$ that will appear as boundaries of half-spaces constituting P will be have $c = a-1$ or $c = N$.
- No vertex of P lies on $Z_{\mathfrak{t}^*}$.
- Given a polytope $P \subseteq {}^b\mathfrak{t}^*$, there is a (classic) polytope $\Delta_Z \subseteq \mathfrak{t}_Z^*$ having the property that the intersection of P with each component of $Z_{{}^b\mathfrak{t}^*}$ is Δ_Z.
- P is locally isomorphic to $\{-\varepsilon \le y_i \le \varepsilon\}\times\Delta_Z \subseteq {}^b\mathbb{R}\times\mathfrak{t}_Z^*$ near each component of $Z_{{}^b\mathfrak{t}^*}$, and is isomorphic to $\Delta_Z\times\mathbb{R}$ in any component $\{i\}\times\mathfrak{t}^*\cong\mathfrak{t}^*$ except $i\in\{a-1,N\}$.
- Any polytope in ${}^b\mathfrak{t}^*$ is isomorphic to $\Delta_Z\times{}^b\mathbb{R}$.
- For $i\in\{a-1,N\}$, the restriction of P to $\{i\}\times\mathfrak{t}^*$ is a polyhedron with recession cone[6] equal to $\mathbb{R}_0^+v$ (if i is even) or $\mathbb{R}_0^-v$ (if i is odd), where v is the distinguished direction in $\mathfrak{t}^*$ used to define ${}^b\mathfrak{t}^*$.

Because no vertex of P lies on $Z_{\mathfrak{t}^*}$, the definition of a Delzant polytope generalizes easily to the context of b-polytopes.

Definition 42. A b-polytope $P\subseteq{}^b\mathfrak{t}^*$ is **Delzant** if for every vertex v of P, there is a lattice basis $\{u_i\}$ of $\mathfrak{t}^*$ such that the edges incident to v can be written near v in the form $v+tu_i$ for $t\ge 0$. A b-polytope $P\subseteq{}^b\mathfrak{t}^*/\langle N\rangle$ (which has no vertices) is **Delzant** if the polytope $\Delta_Z\subseteq\mathfrak{t}_Z^*$ is Delzant.

[6]The recession cone of a convex set A containted in a vector space V is $recc(A) = \{v \in V \mid \forall_{a\in A}\, a+v\in A\}$.

The left polytope in Figure 12 is not Delzant – the Delzant condition is not satisfied at the vertex at the top of the picture in the center column of lattice points. However, the Delzant condition in satisfied at all other vertices. The right polytope in Figure 12 is Delzant. Given a Delzant b-polytope P, the intersection of P with a component of $Z_{{}^b\mathfrak{t}^*}$ (or $Z_{{}^b\mathfrak{t}^*/\langle N\rangle}$) is a Delzant polytope in $\mathfrak{t}_Z^*$. By the properties of b-polytopes, it follows that this Delzant polytope does not depend on which component of $Z_{{}^b\mathfrak{t}^*}$ (or $Z_{{}^b\mathfrak{t}^*/\langle N\rangle}$) is chosen.

Definition 43. Given a b-polytope P, the **extremal polytope** Δ_P is the Delzant polytope in $\mathfrak{t}_Z^*$ given by $P \cap Z'$, where Z' is any connected component of $Z_{{}^b\mathfrak{t}^*}$ (or $Z_{{}^b\mathfrak{t}^*/\langle N\rangle}$).

Before proving the Delzant theorem in our context, we need the following,

Proposition 44. *Let $(X_\Delta, \omega_\Delta, \mathbb{T}^{n-1}, \mu_\Delta : X_\Delta \to \Delta)$ be a (classic) compact connected symplectic toric manifold, and $a < b \in \mathbb{R}$. Consider the non-compact symplectic toric manifold*

$$(M = (a,b) \times \mathbb{S}^1 \times X_\Delta, \omega_M = dy \wedge d\theta + \omega_\Delta, \mathbb{S}^1 \times \mathbb{T}^{n-1}, (y, \mu_\Delta) : (a,b) \times \Delta)$$

where y and θ are the standard coordinates on (a,b) and $\mathbb{S}^1$ respectively. This symplectic toric manifold has the property that any vector field which is both symplectic and tangent to the fibers of the moment map is a Hamiltonian vector field.

Proof. Choose any $y_0 \in (a,b), x_0 \in X_\Delta$, and consider the loop

$$\gamma : \mathbb{S}^1 \to (a,b) \times \mathbb{S}^1 \times X_\Delta, \quad t \mapsto (y_0, t, x_0).$$

Integration of a 1-form on γ represents an element of $H^1(M)^*$ which pairs nontrivially with $[d\theta]$ and hence is itself nontrivial. By the Künneth formula,

$$H^1(M) \cong (H^0((a,b) \times \mathbb{S}^1) \otimes H^1(X_\Delta)) \oplus (H^1((a,b) \times \mathbb{S}^1) \otimes H^0(X_\Delta))$$

which is one-dimensional due to the fact that the cohomology of a compact symplectic toric manifold is supported in even degrees. Therefore, a closed 1-form on M is exact precisely if its integral along γ is zero.

Let v be a symplectic vector field on M tangent to the fibers of the moment map. Because the fibers of the moment map are isotropic and because the image of γ is contained in a single such fiber, it follows that $\omega_M(v, \gamma_*(\partial/\partial t)) = 0$ at all points in the image of γ. Therefore, the integral of $\iota_v\omega$ along γ vanishes, so $\iota_v\omega$ is exact and therefore v is Hamiltonian. $\square$

Theorem 45. *For a fixed primitive lattice vector $v \in \mathfrak{t}^*$ and weight function $wt : [1, N] \to \mathbb{R}_{>0}$, the maps*

$$\left\{ \begin{array}{c} b\text{−}symplectic\ toric\ manifolds \\ (M, Z, \omega, \mu : M \to {}^b\mathfrak{t}^*) \end{array} \right\} \to \left\{ \begin{array}{c} \textit{Delzant } b\textit{-polytopes} \\ \textit{in } {}^b\mathfrak{t}^* \end{array} \right\} \tag{6}$$

and

$$\left\{ \begin{array}{c} b\text{−}symplectic\ toric\ manifolds \\ (M, Z, \omega, \mu : M \to {}^b\mathfrak{t}^*/\langle N\rangle) \end{array} \right\} \to \left\{ \begin{array}{c} \textit{Delzant } b\textit{-polytopes} \\ \textit{in } {}^b\mathfrak{t}^*/\langle N\rangle \end{array} \right\} \tag{7}$$

that send a b-symplectic toric manifold to the image of its moment map are bijections, where the sets on the right should be considered as equivalent up to equivariant b-symplectomorphism that preserves the moment map.

Proof. To prove surjectivity in the ${}^b\mathfrak{t}^*$ case, let P be a Delzant b-polytope, and first construct the (classic) symplectic toric manifold $(X_Z, \omega_Z, \mu_Z : X_Z \to \mathfrak{t}_Z^*)$ associated with the extremal polytope Δ_P. Pick some $X \in \mathfrak{t}$ that pairs positively with the distinguished vector in the definition of ${}^b\mathfrak{t}^*$, which induces an identification ${}^b\mathfrak{t}^* \cong {}^b\mathbb{R} \times \mathfrak{t}_Z^*$. Let I be a closed interval in ${}^b\mathbb{R}$ large enough that $I \times \Delta_{Z_{b_{\mathfrak{t}^*}}} \supseteq P$. Let $(\mathbb{S}^2, Z_S, \omega_S, \mu_S : \mathbb{S}^2 \to {}^b\mathbb{R})$ be a b-symplectic toric manifold having $I \subseteq {}^b\mathbb{R}$ as its moment map image. Then

$$(\mathbb{S}^2 \times X_Z, \omega_S \times \omega_Z, (\mu_S, \mu_Z))$$

is a b-symplectic toric manifold having $I \times \Delta_{Z_{b_{\mathfrak{t}^*}}}$ as the image of its moment map. By performing a sequence of symplectic cuts inside $\{0\} \times \mathfrak{t}^*$ and $\{N-1\} \times \mathfrak{t}^*$, we arrive at a b-symplectic toric manifold having P as its moment map image.

To prove surjectivity in the ${}^b\mathfrak{t}^*/\langle N \rangle$ case, we again begin by constructing the (classic) symplectic toric manifold $(X_Z, \omega_Z, \mu_Z : X_Z \to \mathfrak{t}_Z^*)$ associated with the polytope $\Delta_{Z_{b_{\mathfrak{t}^*}}}$. Pick some $X \in \mathfrak{t}$ that pairs positively with the distinguished vector in the definition of ${}^b\mathfrak{t}^*$, which induces an identification ${}^b\mathfrak{t}^*/\langle N \rangle \cong {}^b\mathbb{R}/\langle N \rangle \times \mathfrak{t}_Z^*$. Let $(\mathbb{T}^2, Z_T, \omega_T, \mu_T : T^2 \to {}^b\mathbb{R}/\langle N \rangle)$ be a b-symplectic toric manifold having all of ${}^b\mathbb{R}/\langle N \rangle$ as its moment map image. Then

$$(\mathbb{T}^2 \times X_Z, \omega_T \times \omega_Z, (\mu_T, \mu_Z))$$

is a b-symplectic toric manifold having P as the image of its moment map.

The proof of injectivity is inspired by the proof of Proposition 6.4 in [LT]. We prove the statement when the adjacency graph is a line; the proof is the same (with different notation) in the case when the adjacency graph is a cycle. Let (M, Z, ω, μ) and (M', Z', ω', μ') be two different b-symplectic toric manifolds having the same moment map image. Pick $X \in \mathfrak{t}$ that pairs positively with the distinguished vector $v \in \mathfrak{t}^*$. For each component $\{i\} \times \mathfrak{t}_Z^*$ of $Z_{{}^b\mathfrak{t}^*}$, by Proposition 32 there is an $\varepsilon_i > 0$ such that there is an equivariant isomorphism $\varphi_{Z_i} : \mu^{-1}(P_{Z_i}) \to \mu'^{-1}(P_{Z_i})$, where

$$P_{Z_i} = \{-\varepsilon \leq y_a \leq \varepsilon\} \times \Delta_Z \subseteq P \subseteq {}^b\mathbb{R} \times \mathfrak{t}_Z^*$$

Similarly, for any $N > 0$, there is an equivariant isomorphism $\varphi_{W_i} : \mu^{-1}(P_{W_i}) \to \mu'^{-1}(P_{W_i})$, where $P_{W_i} = \{i\} \times (-N, N) \times \mathfrak{t}_Z^* \subseteq {}^b\mathfrak{t}^*$. Pick N sufficiently large that the open sets $\{P_{W_i}\} \cup \{P_{Z_i}\}$ cover P: see Figure 13.

If the equivariant isomorphisms φ_{Z_i} and φ_{W_j} agreed on $U_{ij} := \mu^{-1}(P_{W_i} \cap P_{Z_j})$ for all i, j, then we could glue these isomorphisms together and the proof of injectivity would be complete. Therefore, it suffices to show for every U_{ij} that there is an equivariant automorphism ψ_{W_i} of $\mu^{-1}(P_{W_i})$ such that

$$\varphi_{W_i} \circ \psi_{W_i}\big|_{U_{ij}} = \varphi_{Z_j}\big|_{U_{ij}} \quad \text{and} \quad \varphi_{W_i} \circ \psi_{W_i}\big|_{U_{ik}} = \varphi_{W_j}\big|_{U_{ik}}$$

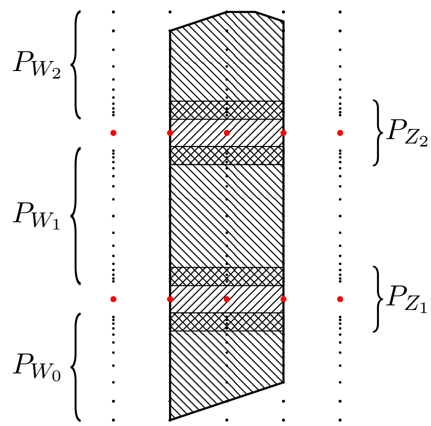


FIGURE 13. The subsets P_{Z_i} and P_{W_i} of a Delzant b-polytope.

for $k \neq j$. Then by replacing φ_{W_i} with $\varphi_{W_i} \circ \psi_{W_i}$, the isomorphisms φ_{Z_i} and φ_{W_j} can be glued. Repeating this process for each U_{ij} gives the desired global equivariant isomorphism.

Let ϕ be the automorphism of U_{ij} given by $\varphi_{W_i}^{-1} \circ \varphi_{Z_j}$. We must extend this automorphism to an automorphism of $\mu^{-1}(P_{W_i})$ which is the identity outside an arbitrarily small neighborhood of U_{ij}. Notice that ϕ is a $\mathbb{T}$-equivariant symplectic diffeomorphism that preserves orbits. Therefore, by Theorem 3.1 in [HS], there exists a smooth $\mathbb{T}$-invariant map $h : U_{ij} \to \mathbb{T}^n$ such that $\phi(x) = h(x) \cdot x$. By the $\mathbb{T}$-invariance of h and the contractibility of $\mu(U_{ij}) = P_{W_i} \cap P_{Z_j}$, there is a map $\theta : U_{ij} \to \mathfrak{t}$ such that $\exp \circ \theta = h$. Define the vector field X_θ to be $X_\theta(x) = \frac{d}{ds}\big|_{s=0} \exp(s\theta(x)) \cdot x$. Observe that X_θ is a symplectic vector field whose time one flow is the symplectomorphism ϕ. By Proposition 44, the vector field is Hamiltonian. Pick an $\hat{f}$ such that $d\hat{f} = \iota_{X_\theta}\omega$. Extend $\hat{f}$ to be a function f on all of $\mu^{-1}(P_{W_i})$ that is locally constant outside a small neighborhood of U_{ij}. Then the time-1 flow of the Hamiltonian vector field corresponding to f will be the desired symplectic automorphism of $\mu^{-1}(P_{W_i})$.

□

Notice that the proof of the surjectivity of the bijections in Theorem 45 is unlike the proof of surjectivity in the classical Delzant theorem, since we do not construct the b-symplectic manifold globally through a symplectic cut in some large $\mathbb{C}^d$. However, we suspect that such a construction is possible by replacing an appropriate direction in $\mathbb{C}^d$ with a suitable b-object, similar to how we replaced a copy of $\mathbb{R} \subseteq \mathfrak{t}^*$ with a copy of ${}^b\mathbb{R}$ in our construction of ${}^b\mathfrak{t}^*$. We invite the interested reader to write down the details.

The moment image of a $2n$-dimensional b-symplectic toric manifold is represented by an n-dimensional polytope P, and the corresponding extremal polytope Δ_P is an $(n-1)$-dimensional Delzant polytope.

For $n = 1$, the extremal polytope is a point, and therefore a b-symplectic toric surface is equivariantly b-symplectomorphic to either a b-symplectic torus $\mathbb{T}^2$ or to a manifold obtained from a b-symplectic sphere $\mathbb{S}^2$ by a series of symplectic cuts away from the exceptional curves that form Z.

For $n = 2$, the extremal polytope is a line segment, corresponding to a symplectic toric sphere. As a consequence, a b-symplectic toric 4-manifold is equivariantly b-symplectomorphic to either a product $\mathbb{T}^2 \times \mathbb{S}^2$ of a b-symplectic torus with a symplectic sphere, or to a manifold obtained from the product $\mathbb{S}^2 \times \mathbb{S}^2$ of two spheres, one b-symplectic and the other symplectic, by a series of symplectic cuts which avoid Z. In particular, $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ can be obtained from a b-symplectic $\mathbb{S}^2 \times \mathbb{S}^2$ with connected Z via symplectic cutting and therefore can be endowed with a b-symplectic toric structure. Because Z was connected (in fact, it would suffice for Z to have an odd number of connected components), there will be fixed points in both the portion of the manifold with positive orientation and in the one with negative orientation. Blowing up these fixed points (each such blow up destroys one fixed point and creates two new ones with the same orientation) corresponds to performing connect sum with either $\mathbb{C}P^2$ or $\overline{\mathbb{C}P^2}$, according to the orientation. Therefore, any $m\mathbb{C}P^2 \# n\overline{\mathbb{C}P^2}$, with $m, n \geq 1$ can be endowed with b-symplectic toric structures (compare with Figure 1 and Corollary 5.2 of [C]).

6. Further directions

6.1. b^2 (b^k) **case.** A section of the b-cotangent bundle is a differential form with a particularly tame order-one singularity along Z. Some differential forms appearing in complex geometry have higher order singularities. For example, $y^{-2}dy \wedge dx$ is the standard differential form on the hyperbolic upper half space $\{y > 0\}$; we can view it as a form on $\mathbb{R}^2$ having a degree-two singularity on the hypersurface $\{y = 0\}$. Towards the goal of studying differential forms with higher order singularities, the author of [Sc] introduces the notions of a b^k**-manifold** and the b^k**-cotangent bundle** for $k \in \mathbb{Z}_{\geq 0}$. A section of this b^k-cotangent bundle is a differential form with a particular kind of degree k singularity along a hypersurface. The precise definitions are technical; details can be found in [Sc]. Here, we give a simple example of a moment map on a b^2 manifold.

Example 46. Consider the differential form $\omega = h^{-2}dh \wedge d\theta$ on $\mathbb{S}^2$, where h, θ are the standard coordinates on $\mathbb{S}^2$. Let $Z = \{h = 0\}$. The $\mathbb{S}^1$-action given by the flow of $-\frac{\partial}{\partial\theta}$ is generated on $\mathbb{S}^2 \backslash Z$ by the Hamiltonian function $-h^{-1}$.

Like in the b-case, there are two main obstacles to the construction of a global moment map: we must enlarge the sheaf $C^\infty(\mathbb{S}^2)$ to include objects

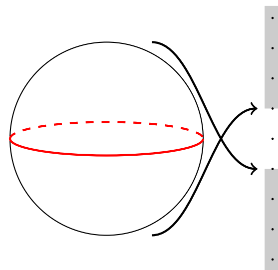


FIGURE 14. The map $-h^{-1}$ on $\mathbb{S}^2 \backslash Z$.

such as $-h^{-1}$, and we must enlarge the codomain so that the map is defined on Z. For this particular example,[7] we define the sheaf of b^2 **functions**

$$^{b^2}C^\infty(\mathbb{S}^2) := \left\{ c_{-1}h^{-1} + c_0 \log|h| + f \;\middle|\; \begin{array}{c} c_{-1}, c_0 \in \mathbb{R} \\ f \in C^\infty(\mathbb{S}^2) \end{array} \right\}$$

and call the $\mathbb{S}^1$-action in our example **Hamiltonian** because it is generated by a b^2 function. Next, we construct an appropriate codomain for our action by identifying[8] the points $(0,\infty)$ and $(1,-\infty)$ in $\{0,1\} \times \overline{\mathbb{R}}$, and then discarding the points $(0,-\infty)$ and $(1,\infty)$. We endow this b^2**-line** with a smooth structure by declaring that the function

$$y_1 : ((0,0),(1,0)) \to \mathbb{R}, \quad y_1 = \begin{cases} -1/x & \text{for points } (0,x) \\ 0 & \text{at } (0,\infty) \\ -1/x & \text{for points } (1,x) \end{cases}$$

is a coordinate function. Then, we can represent the map $-h^{-1}$ as a smooth globally defined moment map $\mu : \mathbb{S}^2 \to {}^{b^2}\mathbb{R}$, which is drawn in Figure 15.

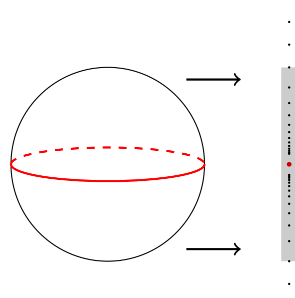


FIGURE 15. A moment map for a toric action on a b^2 manifold.

[7]See [Sc] for the general definition of this sheaf on a general b^k-manifold.

[8]Unlike the construction of the b-line, we glue $+\infty$ to $-\infty$. This reflects the fact that when k is odd (such as when $k = 1$) the singularities of b^k functions approach ∞ (or $-\infty$) on *both* sides of the singularity, but when k is even (such as this example) one side approaches ∞ and one side approaches $-\infty$.

Although the moment map in Figure 15 is visually very similar to Figure 4, we remind the reader that the codomains of these two maps are very different, despite both being homeomorphic to $\mathbb{R}$. Also, to develop the theory of b^k symplectic toric manifolds in its full generality, one would need to assign weights (perhaps even $\mathbb{R}^k$-valued weights) to the components of the codomain "at infinity."

6.2. **Cylindrical moment map.** In classic symplectic geometry, several generalizations of the standard moment map have been studied (Chapter 5 of [OR]). We suspect that many of these generalizations extend to the b-geometry setting as well. One such generalization of the standard moment map, called the *cylinder valued moment map* (introduced in [CDM], an English reference is Section 5.2 of [OR]), is defined for any symplectic Lie group action, even when the action is not Hamiltonian. When the Lie group is a torus and the action is especially well-behaved (specifically, when the holonomy group of a certain connection related to the action is a closed subgroup of $\mathfrak{t}^*$), the cylinder valued moment map enjoys many of the same properties as the standard moment map ([OR], Prop. 5.4.4). In Example 47, we give an example without details of what a cylinder-valued b-moment map might look like.

Example 47. Let $f : \mathbb{S}^1 = \mathbb{R}/2\pi \to \mathbb{R}$ be a smooth nonnegative bump function, supported on $(\pi/4, 3\pi/4)$, with $\int_{\mathbb{S}^1} f(\theta)d\theta = 3$. Consider the b-symplectic manifold

$$(\mathbb{T}^2 = \{(\theta_1, \theta_2) \in (\mathbb{R}/2\pi)^2\}, Z = \{\theta_1 \in \{0, \pi\}\}, \omega = (\csc\theta_1 + f(\theta_1))\, d\theta_1 \wedge d\theta_2)$$

with $\mathbb{S}^1$-action given by the flow of $v = \frac{\partial}{\partial\theta_2}$. The graph of $(\csc\theta_1 + f(\theta_1))$ is shown in Figure 16; observe that ω differs from the b-symplectic form from Example 21 by the presence of f in the formula of ω, which appears as the "bump" in the graph in Figure 16 near $\theta_1 = \pi/2$.

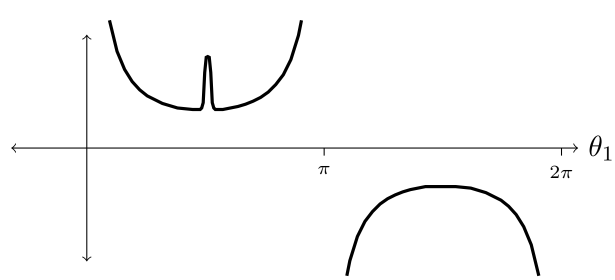


FIGURE 16. The graph of $\frac{1}{\sin\theta_1} + f(\theta_1)$.

Because of this bump, the b-form

$$\iota_v\omega = -\left(\frac{1}{\sin\theta_1} + f(\theta_1)\right) d\theta_1$$

has no ${}^bC^\infty$ primitive and there is no globally-defined function moment map for the action to any ${}^b\mathbb{R}$ or ${}^b\mathbb{R}/\langle N\rangle$. However, if ${}^b\mathbb{R}/(2,-3)$ denotes the quotient of ${}^b\mathbb{R}$ (with weight function $\mathbb{Z} \to \{1\}$) by the $\mathbb{Z}$-action $(a, x) \mapsto (a+2, x-3)$, there *is* a well-defined moment map $\mu : \mathbb{T} \to {}^b\mathbb{R}/(2,-3)$ as shown in Figure 17.

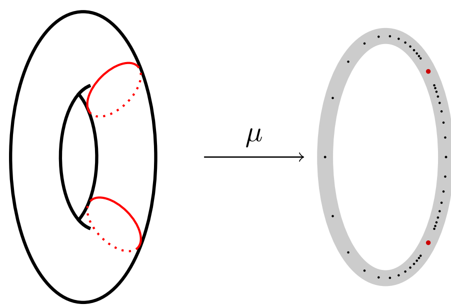


FIGURE 17. A cylindrical moment map.

6.3. **Case of Z self-intersecting transversally.** In most definitions of a b-manifold, Z is required to be an embedded submanifold. However, many of the constructions and results from b-geometry apply even when Z is a union of embedded submanifolds $\{Z_i\}$ which pairwise intersect transversally. To begin with, we can define a bundle ${}^bTM$ over M whose sections are vector fields that are tangent to each Z_i. If $\{x_1, \ldots, x_n\}$ are coordinates for an open $U \subseteq M$ with the property that $Z \cap U = \{x_1 = 0\} \cup \cdots \cup \{x_r = 0\}$, then a trivialization of ${}^bTM$ is given by the sections

$$\left\{ x_1 \frac{\partial}{\partial x_1}, \ldots, x_r \frac{\partial}{\partial x_r}, \frac{\partial}{\partial x_{r+1}}, \ldots, \frac{\partial}{\partial x_n} \right\}.$$

We can generalize the notions of the b-deRham complex, a b-symplectic form, and a b-function in a straightforward manner. Example 48 shows what the moment map might look like for a toric action on one of these objects.

Example 48. If we allow the components of Z to intersect transversally, the following is a b-symplectic manifold

$$(M = \mathbb{S}^2 \times \mathbb{S}^2, Z = \{h_1 = 0\} \cup \{h_2 = 0\}, \omega = \frac{dh_1}{h_1} \wedge d\theta_1 + \frac{dh_2}{h_2} \wedge d\theta_2)$$

where $(h_1, \theta_1, h_2, \theta_2)$ are the standard coordinates on $\mathbb{S}^2 \times \mathbb{S}^2$. The $\mathbb{T}^2$-action

$$(t_1, t_2) \cdot (h_1, \theta_1, h_2, \theta_2) = (h_1, \theta_1 - t_1, h_2, \theta_2 - t_2)$$

is Hamiltonian. Let X_1 and X_2 be the elements of $\mathfrak{t}$ satisfying $X_1^\# = -\frac{\partial}{\partial \theta_1}$ and $X_2^\# = -\frac{\partial}{\partial \theta_2}$ respectively. With weight function $\{0\} \mapsto 1$, there is a smooth moment map

$$M \to {}^b\mathbb{R} \times {}^b\mathbb{R}, \quad (h_1, \theta_1, h_2, \theta_2) \mapsto (\log|h_1|, \log|h_2|),$$

the image of which is illustrated below.

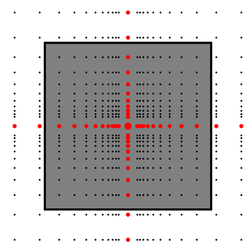


FIGURE 18. A moment map image in $({}^b\mathbb{R})^2$.

6.4. **Integrable systems.** In this section, we present some constructions and results related to integrable systems on b-symplectic manifolds. These results were already announced in [GMP2]. In contrast to the case of Hamiltonian actions on b-manifolds in which we have enlarged the set of admissible functions as components of the moment map to ${}^b\mathcal{C}^\infty(M)$, in this section we will consider integrable systems having as first integrals smooth functions. The reason for considering b-functions in the first place was because they arose naturally as integrals of motion of toric actions (see Proposition 26). If we are interested in the first integrals of the system rather than the associated action, we restrict our choices of first integrals to smooth functions. Throughout this section we will only consider integrable systems of *adapted type*.

Definition 49. An **adapted integrable system** on a b-symplectic manifold (M^{2n}, Z, ω) is a collection of n smooth functions $\{f_1, \dots, f_n\}$ called **first integrals** such that f_1 is a defining function for Z, $\{f_i, f_j\} = 0$ for all i, j, the functions are functionally independent (i.e., $df_1 \wedge \dots df_n \wedge \neq 0$) on a dense set, and the restrictions of $\{f_2, \dots, f_n\}$ to each symplectic leaf of Z are functionally independent on the leaf. The function $\mathbf{F} = (f_1, \dots, f_n)$ is the **moment map** of the integrable system.

This definition is stricter than the more general definition of integrable system on an arbitrary Poisson manifold from [LMV]. The following example is the canonical example of an adapted integrable system on a b-symplectic manifold.

Example 50. Consider $\mathbb{R}^{2n}$ with coordinates $(x_1, y_1, \dots, x_{n-1}, y_{n-1}, t, z)$ and b-symplectic form $\omega_\Pi = \sum_{i=1}^{n-1} dx_i \wedge dy_i + \frac{1}{t}\, dt \wedge dz$. The bivector field corresponding to ω_Π is

$$\Pi = \sum_{i=1}^{n-1} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i} + t\frac{\partial}{\partial t} \wedge \frac{\partial}{\partial z}.$$

The functions $\{x_1, \dots, x_{n-1}, t\}$ define an adapted integrable system with moment map $\mathbf{F} = (x_1, \dots, x_{n-1}, t)$.

The Hamiltonian vector field associated to the defining function of the b-manifold is always a b-vector field which vanishes along the critical hypersurface not only as a section of TM, but also as a section of the b-tangent bundle ${}^bTM$. In Darboux coordinates this vector field is $t\frac{\partial}{\partial z}$. Not all integrable systems (in the sense of [LMV]) on a b-Poisson manifold are adapted. The following is an example of an integrable system which is not adapted.

Example 51. Consider $\mathbb{R}^{2n}$ with coordinates $(x_1, y_1, \dots, x_{n-1}, y_{n-1}, t, z)$ and b-symplectic form $\omega_\Pi = \sum_{i=1}^{n-1} dx_i \wedge dy_i + \frac{1}{t}\, dt \wedge dz$. The functions $\{x_1, \dots, x_{n-1}, z\}$ define an integrable system in the sense of [LMV] with moment map $\mathbf{F} = (x_1, \dots, x_{n-1}, z)$ but this integrable system is not adapted since none of the first integrals is a defining function. Observe that the Hamiltonian vector field associated to the function z is $t\frac{\partial}{\partial t}$ which vanishes as a section of TM but not as a section of ${}^bTM$.

We will now give normal forms for adapted integrable systems on b-symplectic manifolds. Recall the following characterization of the critical hypersurface in the case of compact leaves.

Theorem 52. [GMP2] *If Z is an oriented compact connected regular Poisson manifold of corank one and $\mathcal{F}$ is its symplectic foliation, then $c_{\mathcal{F}} = \sigma_{\mathcal{F}} = 0$ if and only if there exists a Poisson vector field transversal to $\mathcal{F}$. If furthermore $\mathcal{F}$ contains a compact leaf $\mathcal{L}$, then every leaf of $\mathcal{F}$ is symplectomorphic to $\mathcal{L}$, and Z is the total space of a fibration over $\mathbb{S}^1$ which is a mapping torus associated to the symplectomorphism $\phi : \mathcal{L} \to \mathcal{L}$ given by the holonomy map of the fibration over $\mathbb{S}^1$.*

When the critical hypersurface of a b-manifold has compact leaves we can use the of symplectic mapping torus construction in the proof of Theorem 52 to extend a given integrable system $(f_1, \dots, f_{n-1})$ on a compact symplectic leaf $\mathcal{L}$ to the entire critical hypersurface. To extend the functions, we let $\hat{f}_i$ be the unique smooth function on the mapping torus that restricts to f_i on the leaf and satisfies $v(\hat{f}_i) = 0$, where v is the transverse Poisson vector field from the proof of Theorem 52. This Poisson vector field v satisfies the following two conditions: $\alpha(v) = 1$ and $v \in \ker(\beta)$ where α and β are respectively the defining one and two forms of the cosymplectic structure defined on Z.

The $\hat{f}_i$ functions will also Poisson commute because their Hamiltonian vector fields are tangent to the symplectic foliation and the flow of v defines a symplectomorphism from each leaf to every other leaf which preserves these Hamiltonian vector fields. Also observe that the Hamiltonian vector fields associated to the extended functions $\hat{f}_i$ satisfy the relation $[X_{\hat{f}_i}, v] = 0$. This gives a way to construct an adapted integrable systems on this mapping torus.

According to the definition of integrable systems on a regular Poisson manifold provided in [LMV] we need to add an extra function to have an integrable system on the regular corank-one Poisson manifold Z. The first integrals of this new integrable system on the regular codimension one Poisson manifold are $(f_1, \dots, f_{n-1}, w)$ where w is the pullback of some function on $\mathbb{S}^1$ via the mapping torus projection.

We can also extend an integrable system on a critical hypersurface Z of a b-manifold to an adapted integrable system on a neighborhood of Z by extending each of the functions to this neighborhood, then adding a defining function to this new collection of first integrals. More concretely we can adapt Theorem 54 to the case of integrable systems. To do that, we introduce the notion of equivalence of integrable systems.

Definition 53. Two integrable systems on a Poisson manifold are **equivalent** if there exists a Poisson diffeomorphism which takes the orbits of the distribution given by the Hamiltonian vector fields of one integrable system to those of the other. Two integrable systems are **strongly equivalent** if there exists a diffeomorphism which takes a moment map to the other one.

On a regular Poisson manifold any two given regular integrable systems are locally strongly equivalent. On a symplectic manifold an integrable system with non-degenerate singularities are locally equivalent but not necessarily strongly equivalent in a neighborhood of singular point of an integrable system with hyperbolic functions [Mi].

Theorem 54. [GMP2] *Let Π be a regular co-rank one Poisson structure on a compact manifold Z, and $\mathcal{F}$ the induced foliation by symplectic leaves.*

Then $c_{\mathcal{F}} = \sigma_{\mathcal{F}} = 0$ if and only if Z is the exceptional hypersurface of a b-symplectic manifold (M, Z) whose b-symplectic form induces on Z the Poisson structure Π.

Furthermore, two such extensions (M_0, Z) and (M_1, Z) are b-symplectomorphic on a tubular neighborhood of Z if and only if the image of their modular vector class under the map below is the same:

$$H^1_{Poisson}(M) \to H^1_{Poisson}(Z).$$

We recall how the extension was constructed in [GMP2]: the manifold Z is the exceptional hypersurface of a b-symplectic tubular neighborhood $(Z \times (-\varepsilon, \varepsilon), Z)$ with b-symplectic form

$$\omega = \frac{dt}{t} \wedge p^*\alpha + p^*\beta,$$

where p is the projection onto the first coordinate, $t \in (-\varepsilon, \varepsilon)$ and α and β are, respectively, defining one and two-forms of Z. This b-symplectic form induces on Z precisely the given Poisson structure Π.

Now to extend a given integrable system from the critical hypersurface, we first replace the functions $\hat{f}_i$ with $\hat{f}_i \circ p$, and then add a defining function to this set of first integrals. Indeed any adapted integrable system on a b-symplectic manifold will be necessarily of this type because the orbits of the integrable system need to be tangent to the symplectic foliation when restricted to the critical hypersurface since Z is a Poisson submanifold and we can consider the defining function semilocally as one of the first integrals.

Conversely, assume that we are given an adapted integrable system on M defined in a neighborhood of Z, then the restriction of the first integrals to any of the symplectic leaves induces an integrable system on the symplectic leaf because Z is a Poisson submanifold of M and one of the functions of the integrable system vanishes. Let us denote by $\overline{f_1}, \ldots, \overline{f_{n-1}}$ the set of non-vanishing restricted functions. By considering a transversal vector field v satisfying the conditions $\alpha(v) = 1$ and $v \in \ker(\beta)$ for α and β respectively the defining one and two-forms on Z, we can construct an integrable system on the mapping torus by just using the flow of the transverse vector field v. Now this integrable system on Z is equivalent to the initial one because the integrable systems coincide on a symplectic leaf $\mathcal{L}$ and the restriction of the initial integrable system on any other given symplectic leaf needs to be given by the flow of a Poisson vector field transverse to the symplectic foliation which preserves the integrable system (because the integrable system is defined on the mapping torus). The difference of the two vector fields is tangent to the symplectic foliation (because both transverse vector fields define the same class in the first cohomology group) and preserves the orbits of the integrable system. Thus post-composing with the flow of this vector field we obtain the equivalence in the symplectic mapping torus. We can now obtain an integrable system in a neighborhood of Z just by adding a defining function as first integral.

This proves the following,

Theorem 55. *Let Π be a regular co-rank one Poisson structure on a compact manifold Z with vanishing invariants $c_{\mathcal{F}} = \sigma_{\mathcal{F}} = 0$ endowed with an integrable system $\mathbf{F}$, and $\mathcal{F}$ the induced foliation by symplectic leaves. Then the triple $(M, \Pi, \mathbf{F})$ extends semi-locally to a b-symplectic manifold (M, Z) endowed with an adapted integrable system $\tilde{\mathbf{F}}$ obtained by adding the defining function of (M, Z) to the first integrals induced on a symplectic leaf of the foliation $\mathcal{F}$.*

Furthermore, two such extensions $(M_0, Z, \tilde{\mathbf{F}}_1)$ and $(M_1, Z, \tilde{\mathbf{F}}_2)$ are equivalent on a tubular neighborhood of Z if and only if the image of their modular vector class under the map below is the same:

$$H^1_{Poisson}(M) \to H^1_{Poisson}(Z).$$

Observe that as a consequence, this implies that any adapted integrable system is semilocally *split* into a product of an integrable system on a symplectic leaf with an integrable system on the transversal which is 2-dimensional. This is not the case for general Poisson manifolds. The reader is invited to consult [LMV] and [LM] for discussions about splittability of integrable systems in the general Poisson context.

We can use this construction to give normal form results for integrable systems on a b-symplectic manifold. For instance we can obtain normal form results for integrable systems with non-degenerate singularities.

Let us first recall the general statement about normal form results for integrable systems admitting non-degenerate singularities. A singular point of an integrable system is **non-degenerate** if its linear part defines a Cartan subalgebra of $Q(2n,\mathbb{R})$. Cartan subalgebras were classified by Williamson:

Theorem 56. [W] *For any Cartan subalgebra $\mathcal{C}$ of $Q(2n,\mathbb{R})$ there is a symplectic system of coordinates $(x_1, y_1, \ldots, x_n, y_n)$ in $\mathbb{R}^{2n}$ and a basis $h_1, \ldots, h_n$ of $\mathcal{C}$ such that each h_i is one of the following:*

$$\begin{array}{lll} h_i = x_i^2 + y_i^2 & \text{for } 1 \leq i \leq k_e\ , & \text{(elliptic)} \\ h_i = x_i y_i & \text{for } k_e + 1 \leq i \leq k_e + k_h\ , & \text{(hyperbolic)} \\ \begin{cases} h_i = x_i y_i + x_{i+1} y_{i+1}, \\ h_{i+1} = x_i y_{i+1} - x_{i+1} y_i \end{cases} & \begin{array}{c} \text{for } i = k_e + k_h + 2j - 1, \\ 1 \leq j \leq k_f \end{array} & \text{(focus-focus pair)} \end{array}$$

Let $h_1, \ldots, h_n$ be a Williamson basis of this Cartan subalgebra. We denote by X_i the Hamiltonian vector field of h_i with respect to ω and we consider $\mathcal{F}$ the singular foliation defined by the span of these Hamiltonian vector fields. We call $\mathcal{F}$ the **linearized foliation**. It was proved by Eliasson ([E1], [E2]) that one can smoothly take the foliation spanned by the Hamiltonian vector fields of an integrable system with non-degenerate singularities to the linearized foliation. The symplectic linearization of these integrable systems was studied by Eliasson and Miranda: [E1, E2] establishes the result in the case of not admitting hyperbolic coordinates, [Mi] considers the general case of normal forms for integrable systems with non-degenerate singularities.

Theorem 57. [E1, E2],[Mi] *Let ω be a symplectic form defined in a neighborhood U of the origin p for which $\mathcal{F}$ is generically Lagrangian, then there exists a local diffeomorphism $\phi : (U, p) \longrightarrow (\phi(U), p)$ such that ϕ preserves the foliation and $\phi^*(\sum_i dx_i \wedge dy_i) = \omega$, with x_i, y_i local coordinates on $(\phi(U), p)$.*

We can apply this result with the same strategy as in the extension theorem for adapted integrable systems to prove the following:

Theorem 58. *Given an adapted integrable system with non-degenerate singularities on a b-symplectic manifold (M, Z), there exist Eliasson-type normal forms in a neighborhood of points in Z and the minimal rank for these singularities is 1 along Z.*

One may even take these techniques further to give a more general action-angle theorem than the one in [LMV] for b-symplectic manifolds and also consider the classification of semitoric systems [PN1, PN2] in the b-category. We plan to consider these issues in a future paper. Both results rely strongly on the study of toric actions on b-manifolds provided in this paper.

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